VSM COLLEGE OF ENGINEERING

EEE DEPARTMENT

LECTURE NOTES

ON

DIGITAL CONTROL SYSTEMS(R16 reg)

2. *Syllabus*

UNIT – I:
Introduction and signal processing

Introduction to analog and digital control systems – Advantages of digital systems – Typical examples – Signals and processing – Sample and hold devices – Sampling theorem and data reconstruction – Frequency domain characteristics of zero order hold.

UNIT–II:

Z–transformations

Z–Transforms – Theorems – Finding inverse z–transforms – Formulation of difference equations and solving – Block diagram representation – Pulse transfer functions and finding open loop and closed loop responses.

UNIT–III:
State space analysis and the concepts of Controllability and observability
State Space Representation of discrete time systems – State transition matrix and methods of evaluation – Discretization of continuous – Time state equations – Concepts of controllability and observability – Tests (without proof).

UNIT – IV: Stability analysis

Mapping between the S–Plane and the Z–Plane – Primary strips and Complementary Strips – Stability criterion – Modified Routh's stability criterion and jury's stability test.

UNIT – V:

Design of discrete–time control systems by conventional methods

Transient and steady state specifications – Design using frequency response in the w–plane for lag and led compensators – Root locus technique in the z– plane.

UNIT – VI:

State feedback controllers:

Design of state feedback controller through pole placement – Necessary and sufficient conditions – Ackerman's formula.

Text Book:

1. Discrete–Time Control systems – K. Ogata, Pearson Education/PHI, 2nd Edition

Reference Books:

1. Digital Control Systems, Kuo, Oxford University Press, 2nd Edition, 2003.
2. Digital Control and State Variable Methods by M.Gopal, TMH

Unit – I – introduction to signal processing

UNIT – I:
Introduction and signal processing
Introduction to analog and digital control systems – Advantages of digital systems – Typical examples – Signals and processing – Sample and hold devices – Sampling theorem and data reconstruction – Frequency domain characteristics of zero order hold.

Unit Objectives:

After reading this Unit, you should be able to understand:

- To understand the concepts of digital control systems and assemble various components associated with it. Advantages compared to the analog type.

Unit Outcomes:

- The students learn the advantages of discrete time control systems and the “know how” of various associated accessories.

FUNDAMENTALS OF DIGITAL CONTROL SYSTEMS

The analysis of linear control system is based on the fact that the signals at various points in the system are continuous with respect to time. However, in some applications it is convenient to use any one or more control signals at discrete time intervals of time, for example in some industrial process control applications the signal is available only in sampled data form as a sequence of pulses. The control system using one or more signals at discrete time intervals are known as sampled data or digital or discrete time control systems. Digital controllers are used for achieving optimal performance-for example, in the form of maximum productivity, maximum profit, minimum cost or minimum energy use.

Generally, the controllers are used in control system to modify the error signal for better control action. The controllers are classified into two types

1. Analog controllers
2. Digital controllers

Analog controllers:

- These are constructed using analog elements and their i/p and o/p are analog signals, which are continuous function of time.

- Complex, costlier and once fabricated, it is difficult to alter the controllers.

Digital controllers:

- These are constructed using non-programmable devices, microprocessor based systems or computer based systems.
- These are used complex and time shared control functions
- Simple, versatile, programmable, fast acting and less costly.
- It is easy to alter the control functions by modifying the program instructions.

CONTINUOUS TIME VERSUS DISCRETE TIME CONTROL SYSTEMS

If all the system variables of a control system are functions of time, it is termed as a continuous time control system

Ex: The speed control of a d.c motor with tacho-generator feedback.

If one or more system variables of control system are known at a certain discrete time, it is timed as a discrete time control system.

Ex: The microprocessor or computer based system.

The i/p and o/p signals of discrete time systems are digital or discrete, but the i/p and o/p signals of continuous time systems are analog or continuous time signals. Continuous time systems, whose signals are continuous in time, may be described by differential equations where as in discrete time systems the signals are digital signal or sampled data signals may be described by difference equations.

BLOCK-DIAGRAM OF A DIGITAL CONTROL SYSTEM

A control system which uses a digital computer as a controller or compensator is known as digital control system. The advantages of using a digital computer for compensation include: accuracy, reliability, economy and most importantly, flexibility.

A block diagram of a digital control system is shown in the fig.(2.1). The basic elements of the system are shown by the blocks. The controller operation is controlled by clock. The input and output signal in a digital computer will be will be digital signal, but the error signal (input to the controller) to be modified by the controller and the control signal (output of the controller) to drive the plant are analog in nature. Hence a sample and hold circuit and an analog to digital converter (ADC) are provided at the digital computer input. A digital to analog converter (DAC) and a hold circuit are provided at the digital computer output.

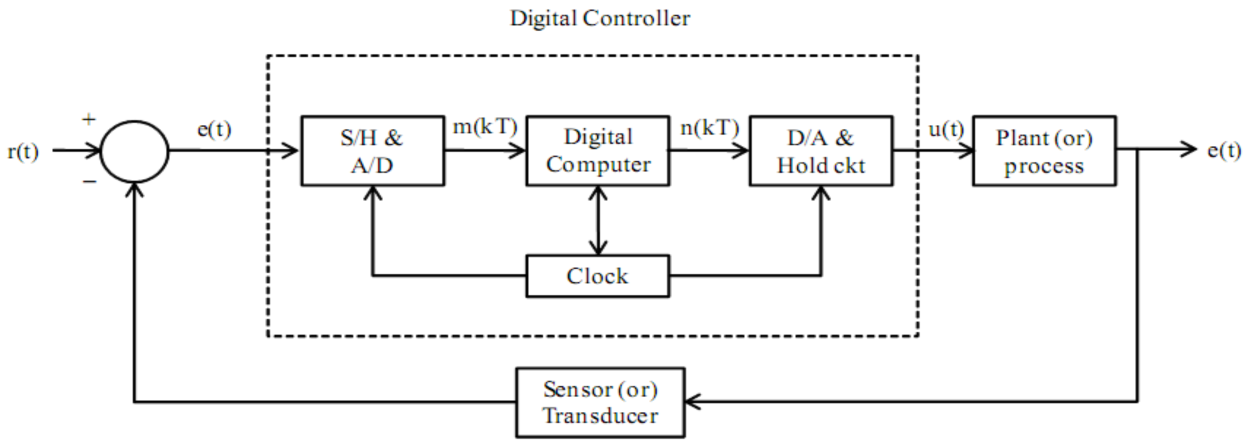


Fig.(2.1): Block-diagram of a digital control system

The sampler (S/H circuit) converts the continuous time error signal into a sequence of pulses and ADC produces binary code (binary number) of each sample. These codes are input data to the digital computer which process the binary code by means of an algorithm and produces another stream of binary codes as output. The DAC and hold circuit converts the output binary code to continuous time signal (analog signal), called control signal is fed to the plant, either directly or through an actuator to drive the plant (or to control its dynamics).

The operation that transforms continuous time signals into discrete-time data is called sampling or Discretization or encoding. The inverse operation, the operation that transform discrete time data into a continuous time signal is called data hold or decoding; it amounts to a reconstruction of a continuous time signal from the sequence of discrete time data. The function of each block in the block diagram is given below

Lecture-2

Sample and hold circuit: Sample and hold circuit is a general term used for sample and hold amplifier. It describes a circuit that receives an analog input signal and holds this signal at a constant value for a specified period of time. Usually the signal is electrical, but other forms are possible such as optical and mechanical.

Analog to digital converter (ADC): An analog to digital converter, also called an encoder, is a device that converts an analog signal into a digital signal, usually a numerically coded signal. Such a converter is needed as an interface between an analog component and a digital component. A sample and hold circuit is often an integral part of a commercially available A/D converter.

The operation of A/D conversion can be explained by the following block diagrams. The

A/D conversion operation is carried out in three stages: sample and hold, quantization and encoding.

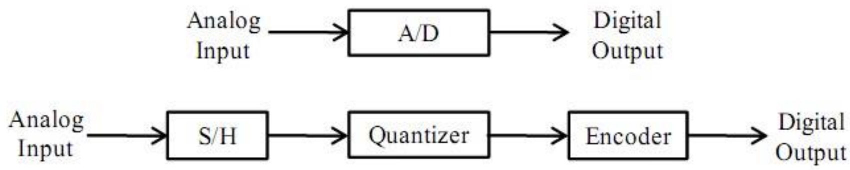


Fig.(2.2): A/D Conversion

Digital to analog converter (DAC): A digital to analog converter, also called as a decoder, is a device that converts a digital signal into an analog signal. Such a converter is needed as an interface between a digital component and an analog component.

The operation of D/A conversion can be explained by the following block diagrams. Two stages are involved in the D/A conversion process: decoding and holding.

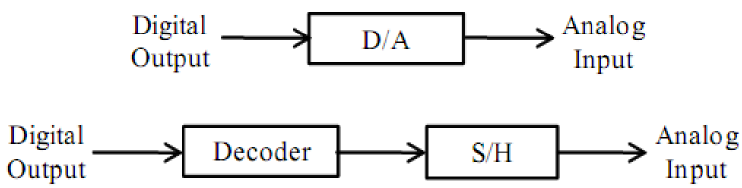


Fig.(2.3): D/A Conversion

Plant or process:A plant is any physical object to be controlled, such as a furnace, a chemical reactor and a set of mechanical parts functioning together to perform a particular operation, such as a servo system or a space craft.

A process is generally defined as a progressive operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead towards a particular result or end.

Transducer: A transducer is a device that converts an input signal into an output signal of another form, such as a device that converts a pressure signal into a voltage output. The output signal in general, depends on past history of the input. Transducers may be classified as analog transducers and sampled-data transducers or digital transducers.

ADVANTAGES & DISADVANTAGES OF DIGITAL CONTROL SYSTEM

Advantages of Digital Control System:

- The advantages of digital control system are listed below:
- Digital components are less susceptible to ageing and environmental variations.
- They are less sensitive to noise and disturbance.

- Digital processors are more compact and light weight.
- They are highly accurate, fast and flexible and more reliable.
- They are growing cheaper in cost.
- Provide high sensitivity to parameter variations.
- Allow more flexibility in programming without an alternation in hardware.
- Digital coded signals can be stored, transmitted, retransmitted, detected, analyzed or processed as desired.
- They are more reliable.

Disadvantages of Digital Control System:

Some of the disadvantages of digital control systems are as follows:

- Conversion of analog signals into discrete time signals and reconstruction introduces noise and errors in the signal.
- Limitations on computing speed and signal resolution.
- Time delays caused in the control loops due to the limitation on computing speed.
- System instability as limit cycles in the closed loop due the finite word length of the processor.

Application of Digital Control System:

Telecommunications

- Multiplexing
- Compression
- Echo control

Audio Processing

- Music
- Speech generation
- Speech recognition

Echo Location

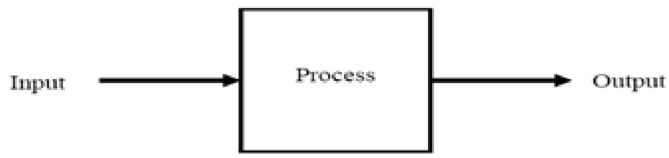
- Radar
- Sonar
- Reflection seismology

Image Processing

- Medical
- Space
- Commercial Imaging Products

EXAMPLES OF DIGITAL CONTROL SYSTEMS

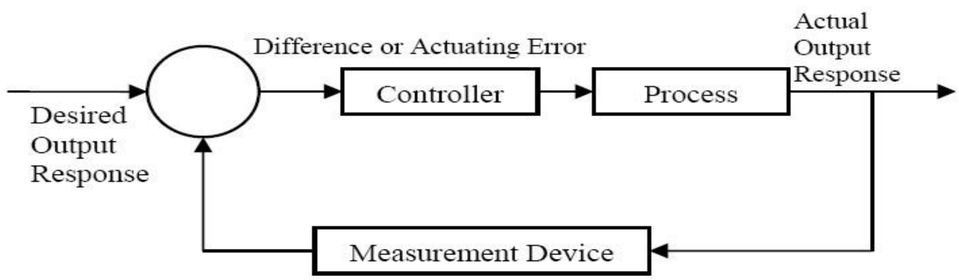
Open-Loop Control System (No feedback)



Feedback

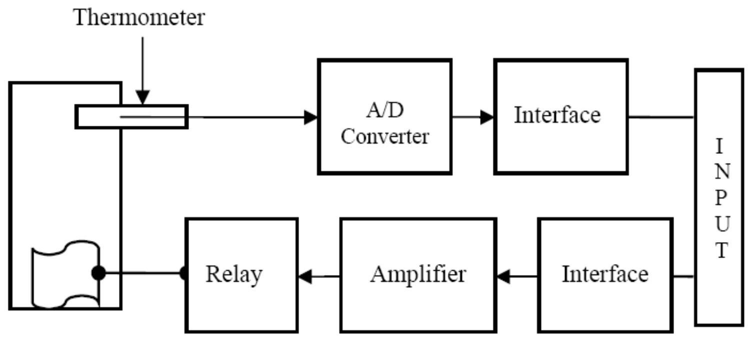
Feedback is a key tool that can be used to modify the behavior of a system. This behavior altering effect of feedback is a key mechanism that control engineers exploit deliberately to achieve the objective of acting on a system to ensure that the desired performance specifications are achieved.

Closed-Loop Control System (with feedback)

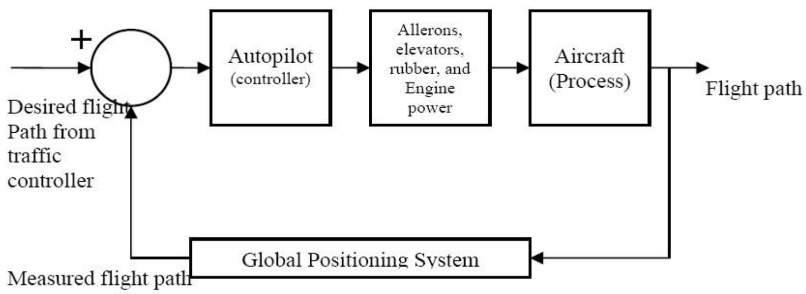


Examples Digital Control System

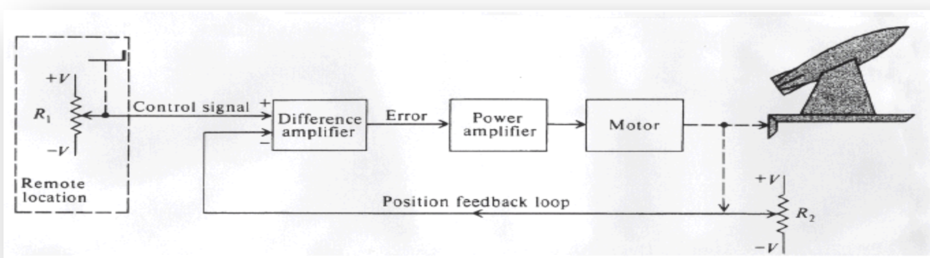
Temperature Control System (Heater or Air Condition):



Autopilot Control System



Missile Launcher



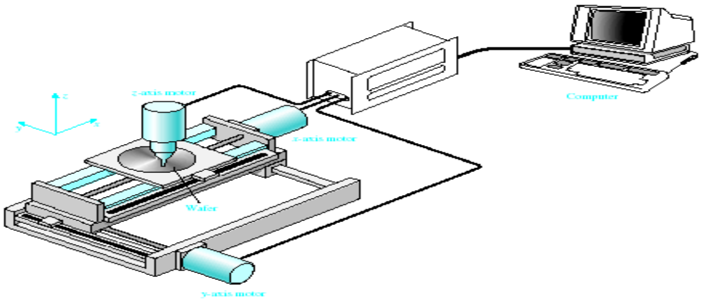
System

(a) Automobile steering control system.

(b) The driver uses the difference between the actual and the desired direction of travel to generate a controlled adjustment of the steering wheel.

(c) Typical direction-of-travel response

A three axis control system for inspecting individual semi conductor wafers with a highly sensitive camera.



Automobile steering control of a Car

SAMPLING PROCESS

Sampling is the conversion of continuous time signal into a discrete time signal obtained by taking samples of continuous time signal at discrete time instants. A sampling process is used whenever a control system involves a digital controller, since a sampling operation and quantization are necessary to enter data into such a controller. Fig.(2.22) shows a switch being used as a sampler. The input to the switch is a continuous time signal denoted by f(t) as shown in fig.2.22(b). The switch is closed for a short duration of time say p, and then remains open for some duration of time. This operation is repeated with ‘T’ called sampling period or sampling interval. The reciprocal of T i.e.Fs =2/T is called sampling rate or sampling frequency. The switch output appears only for the closing duration (p) of switch. The signal f(t) is thus sampled at regular intervals of time as shown in fig.2.22(c).The sampled signal is denoted as f * (t) or f(kT) meaning that f * (t) is obtained after sampling the input signal f(t) at regular intervals of time.

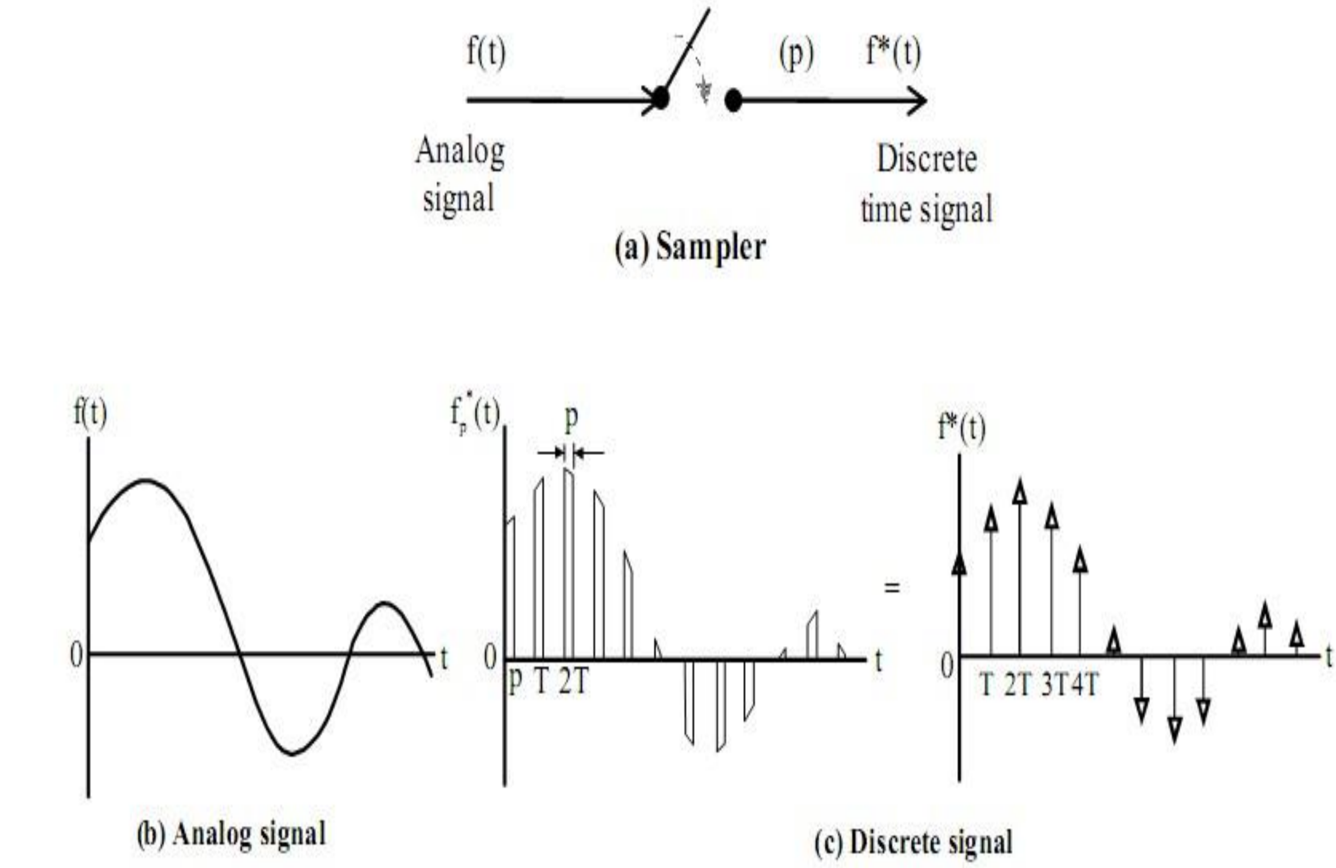


Fig.(2.22): Sampling of an analog signal

A practical sampler acts like a switch closing every T seconds for a short duration of p seconds. Therefore, sampled signal can represented as follows:

$$f_p^*(t) = f(t)\sum_{k=0}^{\infty}\left[u(t-kT) - u(t-kT-p)\right] \quad \text{--- (2.13)}$$

Where u(t) is a unit step function.

The output of an ideal sampler is given by

$$f^*(t) = \sum_{k=0}^{\infty}\left[f(kT)\,\delta(t-kT)\right] \quad \text{--- (2.14)}$$

Applying Laplace transform to the sampled signal (above equation), we have

$$F^*(s) = \sum_{k=0}^{\infty}\left[f(kT)\,e^{-kTs}\right] \quad \text{--- (2.15)}$$

We know from signals and systems classes that the relation between z and Laplace transforms is given by

$$z = e^{sT}$$

$$\Rightarrow s = \frac{1}{T}\ln z$$

$$F^*\left(s = \frac{1}{T}\ln z\right) = F(z) = \sum_{k=0}^{\infty}\left[f(kT)\,z^{-k}\right] \quad \text{--- (2.16)}$$

Where F(z) = Z-transform of $f^*(t) = Z\{r^*(t)\}$

= Laplace transform of $f^*(t)\big|_{s=\frac{1}{T}\ln z}$

TYPES OF SAMPLING OPERATIONS

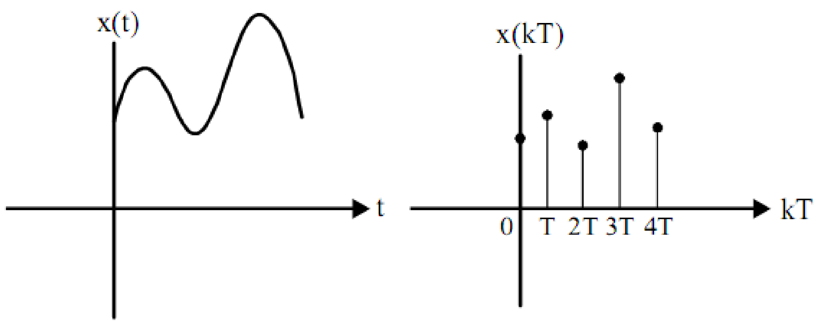
A sampling operation is the process of transforming a continuous time signal into discrete time signal. The different types of sampling operations are:

- **Periodic sampling:** In this case, the sampling instants are equally spaced or t_s = kT (k=0,2,2……….). It is the most conventional type of sampling operation.
- **Multiple order sampling:** The difference between two consecutive sampling instants is repeated periodically. (or) A particular sampling pattern (t_k) is repeated periodically i.e. (t_{k+2}-t_k) is constant for all k.
- **Multiple rate sampling:** In this sampling two simultaneous sampling operations with different time periods carried out on the signal to produce the sampled output. Thus, a digital control system may have different sampling periods in different feedback paths or may have multiple sampling rates

➢ **Random sampling:** In this case, the sampling instants are random or t_k is a random variable.

HOLD DEVICE

The function of hold device is to convert sampled signal into continuous signal. The values of continuous time signal in between the sampling instants are calculated by extrapolation. The sampled signal provided by the sampling process is weighted impulse train of x*(t) which is being the input signal to the system transfer function. The system output is also an impulse train, the envelop of which gives the output, $x^*(t)$ or x(kT) at the sampling instants only as shown in fig.(2.23).The original signal is reconstructed from the sampled signal by using hold circuit.



(a (b)

Fig.(2.23): Sampled data control system- i/p and o/p

The hold circuit brings smoothness in the sampled output. At the sampling instants the hold signal and the original signal have the same value. The use of the hold circuit enables to hold the signal between two consecutive sampling instants at the preceded value till the next sampling instant is reached.

SAMPLING AND HOLDING PROCESS

The basic concept of the sample and hold circuit is shown in figure

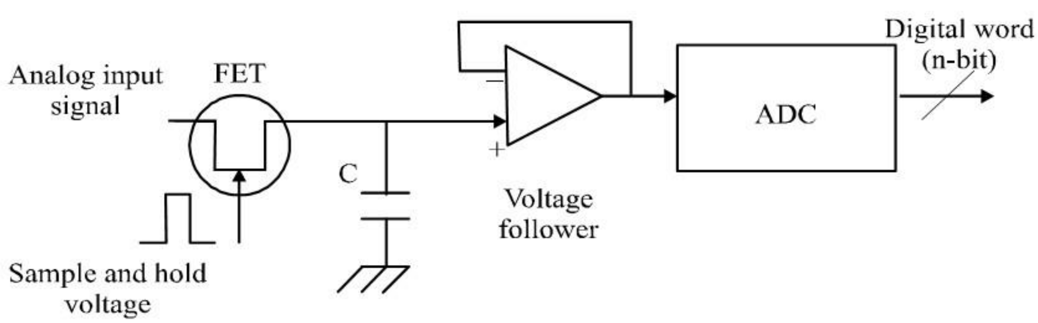


Fig.(2.24): Sample and hold circuit S/H

The sample and hold is connected to the input of ADC. When the electronic switch (FET transistor) is closed the capacitor voltage will track the input voltage. T some time, when a conversion of the input signal is desired, the electronic switch is opened, isolating the capacitor from the input signal. Thus, the capacitor will hold (be charged) to the voltage when the switch was closed. The voltage follower allows this voltage to be impressed upon the ADC input, but the capacitor does not discharge because of very high input impedance of the follower. The start convert is then is then issued, and the conversion proceeds with the input voltage remaining constant from the capacitor. When the conversion is complete the electronic switch is reclosed to capture a new sample and the above sequence is repeated.

Sampling Theorem:

The Sampling Theorem states that "***a signal can be exactly reproduced if it is sampled at a frequency f, where f is greater than twice the maximum frequency in the signal***."

The signals that are used in the real world such as our voices, electrical signals, noise signals etc., are called "analog" signals. To process these signals in computers, we need to convert the signals into "digital" form. An analog signal is continuous in both time and amplitude; where as a digital signal is discrete in both time and amplitude. To convert a signal from continuous time to discrete time, a process called sampling is used. The value of the signal is measured at certain intervals in time and the measurement referred to a sample.

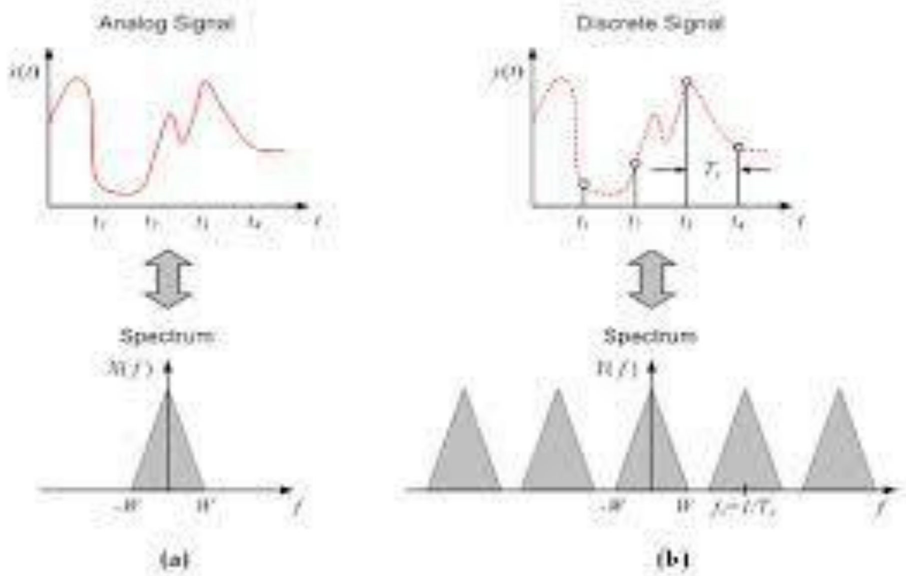
When the continuous analog signal is sampled at a frequency f, the resulting discrete signal has more frequency components than the analog signal. To be precise, the frequency components of the analog signal are repeated at the sample rate, i.e., in the discrete frequency response they are seen at their original position, and are also seen centered around $\pm f$, and around $\pm 2f$ and so on.

If the signal contains high frequency components, we need to sample it at a higher rate to avoid losing information in the signal. In general, to preserve the full information in the signal, it is necessary to sample at twice the maximum frequency of the signal. This is known as the Nyquist rate.

Aliasing Effect:

When the signal is converted back into a continuous time signal, it will exhibit a phenomenon called *aliasing*. Aliasing is the presence of unwanted components in the reconstructed signal. These components were not present when the original signal was sampled. In addition to this some of the frequencies in the original signal may be lost in the reconstructed signal.

Aliasing occurs because signal frequencies can overlap if the sampling frequency is too low. Frequencies "fold" around half the sampling frequency. So this frequency is often referred to as the folding frequency.Sometimes the highest frequency components of a signal are simply noise, or do not contain useful information. To prevent aliasing of these frequencies, we can filter out these components before sampling the signal. Because we are filtering out high frequency components and letting lower frequency components through, this is known as low-pass filtering



ZERO ORDER HOLD (ZOH)

The hold circuit holds the o/p signal at a fixed level between two consecutive sampling instants such that the slope of the hold circuit o/p signal is zero, in other words, hold signal is zeroith derivative of an impulse signal. Such a hold device is called zero order hold (ZOH). It is also known as box-car generator.

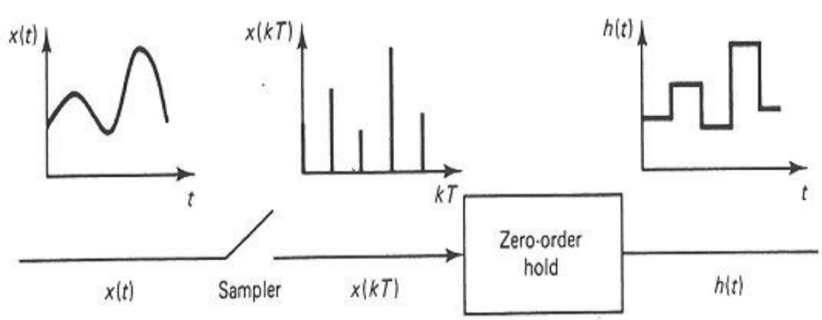


Fig.(2.25)

Transfer Function of ZOH:

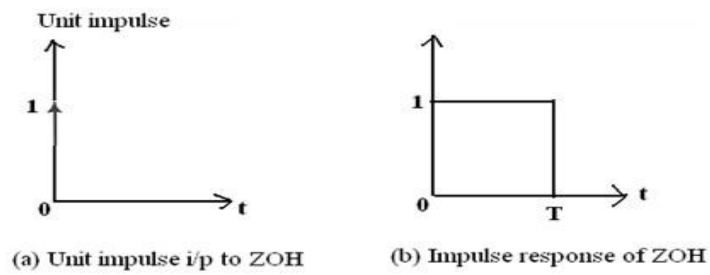


Fig.(2.26)

In Fig.(2.25) shows an unit impulse i/p given to a ZOH circuit, which holds the i/p signal for a duration T and therefore, the o/p appears to be a unit step function till duration T. As the o/p of ZOH is a unit step function appearing up to time T, the same can be written as

h(t)=u(t)-u(t-T)

Taking Laplace transforms on both sides

L[h (t)] =L[u (t)]-L[u (t-T)]

$$H(s) = \frac{1}{s} - \frac{1}{s}e^{-sT}$$

$$H(s) = \frac{1}{s}(1 - e^{-sT})$$

As the input to ZOH is δ(t) i.e. unit impulse function, the Laplace transform of the input is

L[δ(t)]=2

$\therefore$ The transfer function of ZOH is given by

$$G_{h0}(s) = \frac{L[output\ of\ ZOH]}{L[input\ of\ ZOH]} - \frac{1 - e^{-sT}}{s} = \frac{1 - e^{-sT}}{s}$$

$$\Rightarrow G_{h0}(s) = \frac{1}{s}(1 - e^{-sT})$$

Frequency Response Characteristics:

The transfer function of ZOH can be given by

$$G_{h0}(s) = \frac{1}{s}(1 - e^{-sT}) \quad \text{--- (2.19)}$$

To get sinusoidal transfer function, put s=j

$$G_{h0}(j\omega) = \frac{1}{j\omega}(1 - e^{-j\omega T}) \quad \text{--- (2.20)}$$

We know that $e^{-j\omega T/2} . e^{j\omega T/2} = 1$ --- (2.21)

From eqns.(2.26) and (2.27)

$$G_{h0}(j\omega) = \frac{e^{\frac{-j\omega T}{2}} . e^{\frac{j\omega T}{2}} - e^{-j\omega T}}{j\omega}$$

$$= \frac{e^{\frac{-j\omega T}{2}}}{j\omega}[e^{\frac{j\omega T}{2}} - e^{\frac{-j\omega T}{2}}]$$

$$= e^{\frac{-j\omega T}{2}} \times \frac{2}{\omega} \sin\frac{\omega T}{2} \qquad [\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}]$$

$$= T \frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}} \times e^{\frac{-j\omega T}{2}} \quad \text{--- (2.22)}$$

We know that sampling frequency, $\omega_s = \frac{2\pi}{T}$

$$\therefore T = \frac{2\pi}{\omega_s}$$

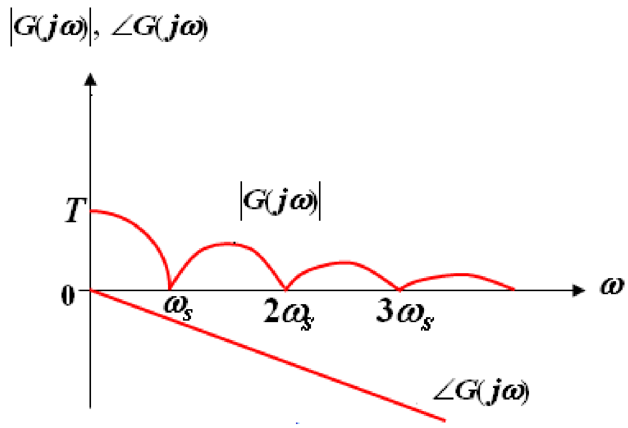
On substituting $T = \frac{2\pi}{\omega_s}$, in eqn.(2.28), we get

$$G_{ho}(j\omega) = \frac{2\pi}{\omega_s} \frac{\sin(\frac{\pi\omega}{\omega_s})}{\frac{\pi\omega}{\omega_s}} \times e^{\frac{-j\pi\omega}{\omega_s}} \quad \text{--- (2.23)}$$

Magnitude of $G_{h0}(j\omega) = |G_{ho}(j\omega)| = \frac{2\pi}{\omega_s} \times \frac{\sin(\frac{\pi\omega}{\omega_s})}{\frac{\pi\omega}{\omega_s}}$ --- (2.24)

Phase angle of $G_{h0}(j\omega) = \angle G_{ho}(j\omega) = -\frac{\pi\omega}{\omega_s}$ --- (2.25)

The frequency response characteristic consists of magnitude response and phase response characteristics, which can be obtained from the eqns. (2.24) and (2.25) respectively. The following fig. (2.27) shows the frequency response curve of ZOH device. From the frequency response curve we can conclude that ZOH device has low-pass filtering



characteristics.

Fig.(2.27): Frequency response of ZOH

Unit – II

Z-Transformations

Z–Transforms – Theorems – Finding inverse z–transforms – Formulation of difference equations and solving – Block diagram representation – Pulse transfer functions and finding open loop and closed loop responses

Unit Objectives:

After reading this Unit, you should be able to understand:

- The theory of z–transformations and application for the mathematical analysis of digital control systems

Unit Outcomes:

- The learner understands z–transformations and their role in the mathematical analysis of different systems (like Laplace transforms in analog systems).

A mathematical tool commonly used for the analysis and synthesis of the discrete time systems is the z-transform. The role of the z-transform in discrete time systems is similar to that of the Laplace transform in continuous time systems. The z-transform provides a method for the analysis of the discrete time systems in the frequency domain than its time domain analysis.

In a linear discrete time control system, a linear difference equation characterizes the dynamics of the system. To determine the system's response to a given input, such a difference equation must be solved. With the z-transform method, the solution's to the linear difference equation become algebraic in nature.

3.2. THE Z-TRANSFORM

The z-transform method is an operational method that is very powerful when working with discrete time systems. For a given sequence values of f(kT), its z-transform is defined by F(z) and is given by

$$F(z) = Z[f(kT)] = \sum_{k=-\infty}^{\infty} f(kT) z^{-k} \qquad \text{--- (3.1)}$$

Where 'z' is a complex variable.

The sequence of the above equation is considered to be two sided and the transform is called two-sided z-transform, since the time index k is defined for both positive and negative values. If the sequence f(kT) is one sided sequence [i.e. f(kT) is defined only for the positive values of k] than the z-transform is called one sided z-transform.

The one sided z-transform of the f(kT) is defined as

$$F(z) = Z[f(kT)] = \sum_{k=0}^{\infty} f(kT) z^{-k} \qquad \text{--- (3.2)}$$

3.3. Z-TRANSFORMS OF SOME STANDARD FUNCTIONS

(i) Unit step function: Consider the unit step function

$$f(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$F(z) = Z[1] = \sum_{k=0}^{\infty} 1 \times z^{-k}$$

$$= 1 + z^{-1} + z^{-2} + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

i.e. if f(t)=1, then F(z)= $\frac{z}{z-1}$

(ii) Unit ramp function: Consider the unit ramp function

$$f(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Put t=kT, k= 0, 1, 2,………

$$F[z] = Z[kT] = \sum_{k=0}^{\infty} (kT)\ ^{-k} = \mathrm{T} \sum_{k=0}^{\infty} kz^{-k}$$

$$= \mathrm{T}[\, z^{-1} + 2z^{-2} + 3z^{-3} + \ldots\ldots\ldots\ldots\ldots\ldots]$$

We know that

$$F(z) = Z[1] = [1 + z^{-1} + z^{-2} + \ldots\ldots\ldots\ldots\ldots\ldots] = \frac{z}{z-1} \qquad \text{--- (3.3)}$$

Differentiating both sides w.r.t 'z'

$$-z^{-2} - 2z^{-3} - 3z^{-4} - \ldots\ldots\ldots\ldots\ldots\ldots = \frac{(z+1)-z}{(z-1)^2} = -\frac{1}{(z-1)^2}$$

$$\Rightarrow z^{-1} + 2z^{-2} + 3z^{-3} + \ldots\ldots\ldots\ldots\ldots\ldots = \frac{z}{(z-1)^2} \qquad \text{--- (3.4)}$$

From eqns.(3.3) and (3.4), we have

$$F(z) = T \times \left[\frac{z}{(z-1)^2}\right] = \frac{zT}{(z-1)^2}$$

i.e., if $f(kT) = kT$, then $F(z) = \dfrac{zT}{(z-1)^2}$

(iii) Exponential function: Consider the exponential function

$$f(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Put t=kT, k= 0, 1, 2,………

$f(kT) = e^{-akT}$, k=0,1,2…….

$$F(z) = Z[e^{at}] = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{-akT} z^{-k}$$

$$= 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \dots\dots\dots$$

$$= 1 + [e^{-at} z^{-1}]^1 + [e^{-at} z^{-1}]^2 + \dots\dots\dots\dots\dots$$

$$= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}$$

i.e., if $f(t) = e^{-at}$ then $F(z) = \frac{z}{z - e^{-aT}}$

Note: Sum of the infinite series, $1 + + r^2 + r^3 + r^4 + \dots\dots = \frac{1}{1-r}$

(iv) Polynomial function, a^t : Let us obtain the z-transform of f(t) as defined by

$$f(t) = \begin{cases} a^t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Put t=kT, k= 0, 1, 2,………

$f(kT) = a^{kT}$

$$F(z) = Z[a^{kT}] = \sum_{k=0}^{\infty} a^{kT} z^{-k}$$

$$= 1 + a^T z^{-1} + a^{2T} z^{-2} + a^{3T} z^{-3} + \dots\dots\dots$$

$$= 1 + (a^T z^{-1})^1 + (a^T z^{-1})^2 + \dots\dots\dots\dots$$

$$= \frac{1}{1 - a^T Z^{-1}} = \frac{z}{z - a^T}$$

i.e., if $f(t) = a^t$, then $F(z) = \frac{z}{z - a^T}$

Noting that

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

we have,

Put t=kT, k=0,1,2…..

$f(kT) = \sin \omega kT$ for k=0,1,2…..

$$F(z) = Z[\sin \omega t] = Z[\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})] = \frac{1}{2j}[\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}]$$

$$= \frac{z}{2j}[\frac{z - e^{-j\omega T} - z + e^{j\omega T}}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1}] = \frac{z}{2j} \times \frac{2j \sin \omega T}{z^2 - 2z\cos \omega T + 1}$$

$$= \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$$

If $f(t) = \sin \omega t$, then $F(z) = \dfrac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$

(ii) Cosine function: consider the cosine function

$$f(t) = \begin{cases} \cos \omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Put t=kT, k=0,1,2…..

$f(kT) = \cos \omega kT$, k=0,1,2,3.....

$$X(z) = Z[\cos \omega t] = Z\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] = \frac{1}{2}[\frac{z}{z - e^{j\omega t}} + \frac{z}{z - e^{-j\omega t}}]$$

$$= \frac{z}{2}[\frac{z - e^{-j\omega T} + z - e^{j\omega T}}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1}] = \frac{z}{2}[\frac{2z - e^{j\omega T} - e^{-j\omega T}}{z^2 - 2z\cos \omega T + 1}]$$

$$= \frac{z(z - \cos \omega T)}{z^2 - 2z\cos \omega T + 1}$$

i.e. if $f(t) = \cos \omega t$, then $F(z) = \dfrac{z(z - \cos \omega T)}{z^2 - 2z\cos \omega T + 1}$

(iii) f(k) = k^P, P being a positive integer

$$= z\sum_{k=0}^{\infty} k^{P-1} k z^{-(k+1)} \quad \text{--- (3.5)}$$

Also by definition

$$Z[k^{P-1}] = \sum_{k=0}^{\infty} k^{P-1} z^{-k} \quad \text{--- (3.6)}$$

Differentiating of eqn.(3.6) with respect to 'z'

$$\frac{d}{dz}\{Z[k^{P-1}]\} = \sum_{k=0}^{\infty} k^{P-1}(-k)^{-(k+1)} \quad \text{--- (3.7)}$$

From eqns. (3.5) and (3.7)

$$Z[k^P] = -z\frac{d}{dz}\{Z[k^{P-1}]\} \quad \text{--- (3.8)}$$

This gives a recurrence formula.

Inverse Z-transforms

Single sided Laplace transform and its inverse make a unique pair, i.e., if *F*(s) is the Laplace transform of *f*(t), then *f*(t) is the inverse Laplace transform of *F*(s). But the same is not true for Z-transform. Say *f*(t) is the continuous time function whose Z-transform is *F*(z). Then the inverse transform is not necessarily equal to *f*(t), rather it is equal to *f*(*k*T) which is equal to *f*(t) only at the sampling instants. Once *f*(t) is sampled by an the ideal sampler, the information between the sampling instants is totally lost and we cannot recover actual *f*(t) from *F*(z).

$$\Rightarrow f(kT) = Z^{-1}[F(z)]$$

The transform can be obtained by using

1. Partial fraction expansion

2. Power series

3. Inverse formula.

The Inverse Z-transform formula is given as:

$$f(kT) = \frac{1}{2\pi j} \oint_{\Gamma} F(z) z^{k-1} dz$$

2.4 Other Z-transform properties

Partial differentiation theorem:

$$Z\left[\frac{\partial}{\partial a}[f(t,a)]\right] = \frac{\partial}{\partial a} F(z,a)$$

Real convolution theorem:

If f_1(t) and f_2(t) have z-transforms F_1(z) and F_2(z) and f_1(t) = 0 = f_2(t) for $t < 0$, then

$$F_1(z)F_2(z) = Z\left[\sum_{n=0}^{k} f_1(nT) f_2(kT - nT)\right]$$

Complex convolution:

$$Z[f_1(t)f_2(t)] = \frac{1}{2\pi j} \oint_{\Gamma} \frac{F_1(\xi)F_2(z\xi^{-1})}{\xi} d\xi$$

Γ : circle / closed path in z-plane which lie in the region $\sigma < |\xi| < \frac{|z|}{\sigma_2}$

σ_1 : radius of convergence of $F_1(\xi)$

σ_1 : radius of convergence of $F_2(\xi)$

- ***2.5 Limitation of Z-transform method***

Ideal sampler assumption
$\Rightarrow$ z-transform represents the function only at sampling instants.

Non uniqueness of z-transform.

Accuracy depends on the magnitude of the sampling frequency w_s relative to the highest frequency component contained in the function *f*(t).

A good approximation of *f*(t) can only be interpolated from *f*(*k*T), the inverse z-transform of*F*(z), by connecting *f*(*k*T) with a smooth curve.

- ***Application of Z-transform in solving Difference Equation***

One of the most important applications of Z-transform is in the solution of linear difference equations. Let us consider that a discrete time system is described by the following difference equation.

$$y(k+2) + 0.5y(k+1) + 0.06y(k) = -(0.5)^{k+1}$$

The initial conditions are *y*(0) = 0, *y*(1) = 0 .
We have to find the solution *y(k)* for *k* > 0.

Taking z-transform on both sides of the above equation:

$$\begin{aligned} z^2Y(z) + 0.5zY(z) + 0.06Y(z) &= -0.5\frac{z}{z-0.5} \\ \text{or,}\ \ Y(z) &= -\frac{0.5z}{(z-0.5)(z^2+0.5z+0.06)} \\ &= -\frac{0.5z}{(z-0.5)(z+0.2)(z+0.3)} \end{aligned}$$

Using partial fraction expansion:

$$Y(z) = -\frac{0.8937z}{z-0.5} + \frac{7.143z}{z+0.2} - \frac{6.25z}{z+0.3}$$

Taking Inverse Laplace: $y(k) = -0.893(0.5)^k + 7.143(-0.2)^k - 6.25(-0.3)^k$

To emphasize the fact that *y(k)* = 0 for *k* < 0 , it is a common practice to write the solution as:

$$y(k) = -0.893(0.5)^k u_s(k) + 7.143(-0.2)^k u_s(k) - 6.25(-0.3)^k u_s(k)$$

where $u_s(k)$ is the unit step sequence.

Example 2:

Find the solution of

$$y(k+2) - 3y(k+1) + 2y(k) = r(k)$$

where $r(k) = 3^k$; $y(0) = 0$ and $y(1) = 1$.

Solution:

The given equation can be written as

$$y(k+2) = 3y(k+1) - 2y(k) + r(k)$$

Taking z-transform

$$z^2Y(z) - z^2Y(0) - zY(1) = 3\big(zY(z) - zY(0)\big) - 2Y(z) + R(z)$$

$$z^2Y(z) = z + 3zY(z) - 2Y(z) + R(z)$$

$$Y(z) = \frac{z + R(z)}{z^2 - 3z + 2}$$

$$= \frac{z + R(z)}{(z-1)(z-2)}$$

$$= \frac{z + \frac{z}{(z-3)}}{(z-1)(z-2)}$$

$$= \frac{z(z-3) + z}{(z-1)(z-2)(z-3)}$$

$$= \frac{z(z-3+1)}{(z-1)(z-2)(z-3)}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-1)(z-3)}$$

Pluse Transfer Function

Transfer function of an LTI (Linear Time Invariant) continuous time system is defined as

$$G(s) = \frac{C(s)}{R(s)}$$

where ***R(s)*** and ***C(s)*** are Laplace transforms of input ***r(t)*** and output ***c(t)***. We assume that initial condition are zero.

Pulse transfer function relates Z-transform of the output at the sampling instants to the Z-transform of the sampled input.

When the same system is subject to a sampled data or digital signal ***r*(t)***, the corresponding block diagram is given in Figure 1 .

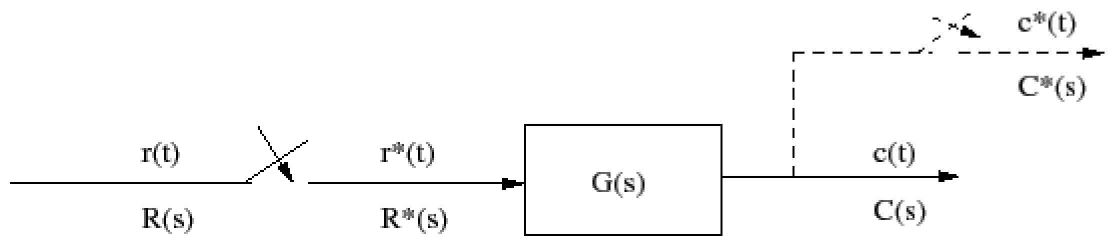


Figure 1: Block diagram of a system subject to a sampled input

The output of the system is ***C(s)*** = ***G(s)R*(s)***. The transfer function of the above system is difficult to manipulates because it contains a mixture of analog and digital components. Thus, for ease of manipulation, it is desirable to express the system characteristics by a transfer function that relates ***r*(t)*** to ***c*(t)***, a fictitious sampler output, as shown in Figure 1.

One can then write:

$$C^*(s) = \sum_{k=0}^{\infty} c(kT)e^{-kTs}$$

Since ***c(kT)*** is periodic,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + jnw_s)$$

with ***c*(0) = 0**

The detailed derivation of the above expression is omitted. Similarly,

$$R^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s + jnw_s)$$

Again,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + jnw_s)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s + jnw_s)G(s + jnw_s)$$

Since ***R*(s)*** is periodic ***R*(s + jnw$_s$)*** = ***R*(s)***. Thus

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s)G(s + jnw_s)$$

$$= R^*(s)\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jnw_s)$$

If we define

$$G^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jnw_s)$$

, then $C^*(s) = R^*(s)G^*(s)$.

$$G^*(s) = \frac{C^*(s)}{R^*(s)}$$

is known as **pulse transfer function**. Sometimes it is also referred to as the **starred transfer function**.

If we now substitute **z = e^{Ts}** in the previous expression, we will directly get the **z-transfer function*G(z)*** as

$$G(z) = \frac{C(z)}{R(z)}$$

G(z) can also be defined as

$$G(z) = \sum_{k=0}^{\infty} g(kT) z^{-k}$$

where ***g(kT)*** denotes the sequence of the impulse response ***g(t)*** of the system of transfer function***G(s)***. The sequence ***g(kT)***, ***k*** = **0, 1, 2,...** , is also known as impulse sequence.

Overall Conclusion

1. Pulse transfer function or Z transfer function characterizes the discrete data system only at sampling instants. The output information between the sampling instants is lost.

2. Since the input of discrete data system is described by output of the sampler, for all practical purposes, the samplers can be simply ignored and the input can be regarded as ***r*(t)***.

Alternate way to arrive at $G(z) = \dfrac{C(z)}{R(z)}$:

$$c^*(t) = g^*(t)\Big|_{\text{when } r^*(t) \text{ is an impulse function}}$$

$$= \sum_{k=0}^{\infty} g(kT)\delta(t - kT)$$

When the input is ***r*(t)***,

$$c(t) = r(0)g(t) + r(T)g(t - T) + ...$$

$$\Rightarrow c(kT) = r(0)g(kT) + r(T)g((k-1)T) + ...$$

$$\Rightarrow c(kT) = \sum_{n=0}^{k} r(nT)g(kT - nT)$$

$$\Rightarrow C(z) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{k} r(nT)g(kT - nT)z^{-k}$$

Using real convolution theorem

$$C(z) = R(z)G(z)$$

$$\Rightarrow G(z) = \frac{C(z)}{R(z)}$$

Pulse transfer of discrete data systems with cascaded elements

Care must be taken when the discrete data system has cascaded elements. Following two cases will be considered here.

- 1. Cascaded elements are separated by a sampler
- 2. Cascaded elements are not separated by a sampler

The block diagram for the first case is shown in Figure 2.

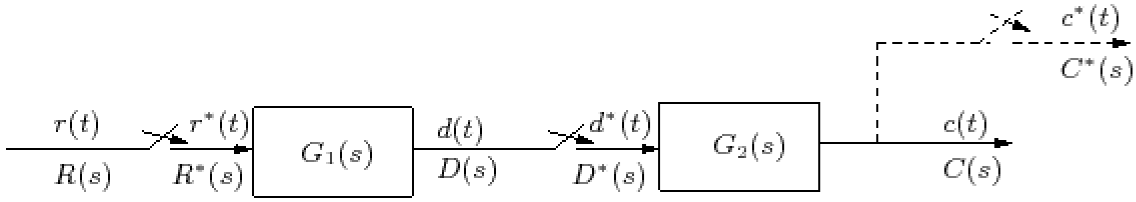


Figure 2: Discrete data system with cascaded elements, separated by a sampler

The input-output relations of the two systems G_1 and G_2 are described by

$$D(z) = G_1(z)R(z)$$

and

$$C(z) = G_2(z)D(z)$$

Thus the input-output relation of the overall system is

$$C(z) = G_1(z)G_2(z)R(z)$$

We can therefore conclude that the z-transfer function of two linear system separated by a sampler are the products of the individual z-transfer functions.

Figure 3 shows the block diagram for the second case.

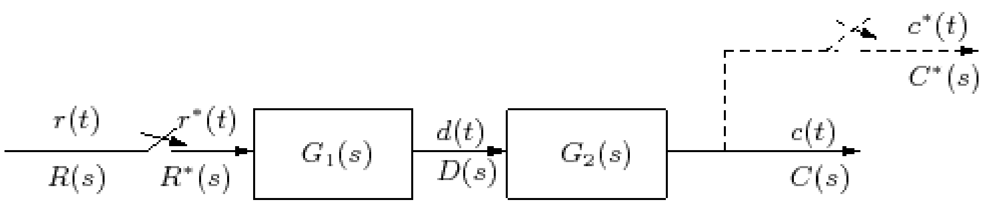


Figure 3: Discrete data system with cascaded elements, not separated by a sampler

The continuous output $C(s)$ can be written as

$$C(s) = G_1(s)G_2(s)R^*(s)$$

The output of the fictitious sampler is

$$C(z) = Z\left[G_1(s)G_2(s)\right]R(z)$$

z-transform of the product $G_1(s)G_2(s)$ is denoted as

$$Z\left[G_1(s)G_2(s)\right] = G_1G_2(z) = G_2G_1(z)$$

One should note that in general $G_1G_2(z) \neq G_1(z)G_2(z)$, except for some special cases. The overall output is thus,

$$C(z) = G_1G_2(z)R(z)$$

Pluse transfer function of ZOH

As derived in lecture 4 of module 1, transfer function of zero order hold is

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s}$$

$\Rightarrow$ Pulse transfer function $G_{ho}(z) = Z\left[\frac{1 - e^{-Ts}}{s}\right]$

$$= (1 - z^{-1})Z\left[\frac{1}{s}\right]$$

$$= (1 - z^{-1})\frac{z}{z-1}$$

$$= 1$$

This result is expected because zero order hold simply holds the discrete signal for one sampling period, thus taking Z-transform of ZOH would revert back its original sampled signal.

A common situation in discrete data system is that a sample and hold (S/H) device precedes a linear system with transfer function ***G(s)*** as shown in Figure 2. We are interested in finding the transform relation between ***r*(t)*** and ***c*(t)***.

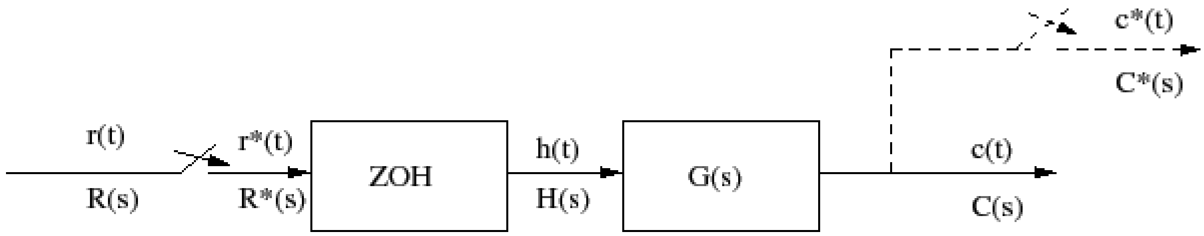


Figure 2: Block diagram of a system subject to a sample and hold process

Z-transform of output ***c(t)*** is

$$
\begin{aligned}
C(z) &= Z\left[G_{ho}(s)G(s)\right]R(z) \\
&= Z\left[\frac{1-e^{-Ts}}{s}G(s)\right]R(z) \\
&= (1-z^{-1})Z\left[\frac{G(s)}{s}\right]R(z)
\end{aligned}
$$

where $(1-z^{-1})Z\left[\frac{G(s)}{s}\right]$ is the Z-transfer function of an S/H device and a linear system.

It was mentioned earlier that when sampling frequency reaches infinity, a discrete data system may be regarded as a continuous data system. However, this does not mean that if the signal ***r(t)*** is sampled by an ideal sampler then ***r*(t)*** can be reverted to ***r(t)*** by setting the sampling time ***T*** to zero. This simply bunches all the samples together. Rather, if the output of the sampled signal is passed through a hold device then setting the sampling time ***T*** to zero the original signal ***r(t)*** can be recovered. In relation with Figure 2,

$$\lim_{T\to 0} H(s) = R(s)$$

Example

Consider that the input is $r(t) = e^{-at}u_s(t)$, where $u_s(t)$ is the unit step function.

$$\Rightarrow R(s) = \frac{1}{s+a}$$

Laplace transform of sampled signal ***r*(t)*** is

$$R^*(s) = \frac{e^{Ts}}{e^{Ts} - e^{-aT}}$$

Laplace transform of the output after the ZOH is

$$\begin{aligned} H(s) &= G_{ho}(s)R^*(s) \\ &= \frac{1-e^{-Ts}}{s} \cdot \frac{e^{Ts}}{e^{Ts} - e^{-aT}} \end{aligned}$$

When $\boldsymbol{T \to 0}$,

$$\lim_{T \to 0} H(s) = \lim_{T \to 0} \frac{1-e^{-Ts}}{s} \frac{e^{Ts}}{e^{Ts} - e^{-aT}}$$

The limit can be calculated using L' hospital's rule. It says that:

If $\lim_{x \to a} f(x) = 0/\infty$ *and if* $\lim_{x \to a} g(x) = 0/\infty$ *, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

For the given example, $\boldsymbol{x = T}$, $f(T) = \frac{1-e^{-Ts}}{s}$ and $g(T) = \frac{e^{Ts} - e^{-aT}}{e^{Ts}}$. Both the expressions approach zero as $\boldsymbol{T \to 0}$. So,

$$H(s) = \lim_{T \to 0} \frac{f(T)}{g(T)}$$

$$= \lim_{T \to 0} \frac{f'(T)}{g'(T)}$$

$$= \lim_{T \to 0} \frac{e^{Ts}}{(s+a)e^{-T(s+a)}}$$

$$= \frac{1}{s+a}$$

$$= R(s)$$

which implies that the original signal can be recovered from the output of the **sample and hold device** if the sampling period approaches zero.

Pluse Transfer Function of Closed Loop Systems

We know that various advantages of feedback make most of the control systems closed loop in nature. A simple single loop system with a sampler in the forward path is shown in Figure 1.

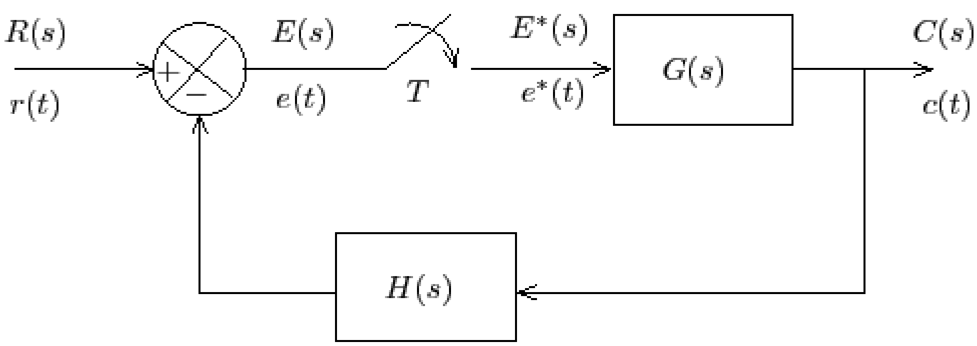


Figure 1: Block diagram of a closed loop system with a sampler in the forward path

The objective is to establish the input-output relationship. For the above system, the output of the sampler is regarded as an input to the system. The input to the sampler is regarded as another output. Thus the input-output relations can be formulated as

$$E(s) = R(s) - G(s)H(s)E^*(s) \tag{1}$$

$$C(s) = G(s)E^*(s) \tag{2}$$

Taking pulse transform on both sides of (1),

$$E^*(s) = R^*(s) - GH^*(s)E^*(s) \tag{3}$$

where

$$GH^*(s) = [G(s)H(s)]^*$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jnw_s)H(s+jnw_s)$$

We can write from equation (3),

$$E^*(s) = \frac{R^*(s)}{1+GH^*(s)}$$

$$\Rightarrow C(s) = G(s)E^*(s)$$

$$= \frac{G(s)R^*(s)}{1+GH^*(s)}$$

Taking pulse transformation on both sides of (2)

$$C^*(s) = [G(s)E^*(s)]^*$$

$$= G^*(s)E^*(s)$$

$$= \frac{G^*(s)R^*(s)}{1+GH^*(s)}$$

$$\therefore \frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1+GH^*(s)}$$

$$\Rightarrow \frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

where $GH(z) = Z[G(s)H(s)]$.

Now, if we place the sampler in the feedback path, the block diagram will look like the Figure 2.

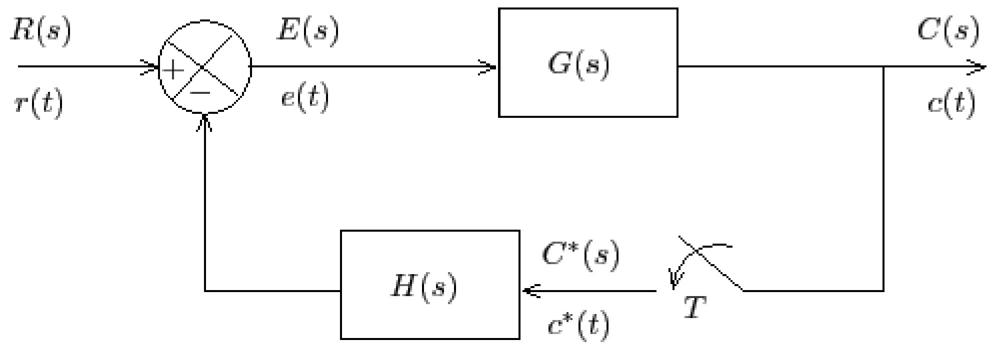


Figure 2: Block diagram of a closed loop system with a sampler in the feedback path

The corresponding input output relations can be written as:

$$E(s) = R(s) - H(S)C^*(s) \tag{4}$$

$$C(s) = G(s)E(s) = G(s)R(s) - G(s)H(s)C^*(s) \tag{5}$$

Taking pulse transformation of equations (4) and (5)

$$E^*(s) = R^*(s) - H^*(s)C^*(s)$$

$$C^*(s) = GR^*(s) - GH^*(s)C^*(s)$$

where,

$$GR^*(s) = [G(s)R(s)]^*$$

$$GH^*(s) = [G(s)H(s)]^*$$

can be written as

$$C^*(s) = \frac{GR^*(s)}{1 + GH^*(s)}$$

$$\Rightarrow C(z) = \frac{GR(z)}{1 + GH(z)}$$

We can no longer define the input output transfer function of this system by either $\frac{C^*(s)}{R^*(s)}$ or $\frac{C(z)}{R(z)}$. Since the input $r(t)$ is not sampled, the sampled signal $r^*(t)$ does not exist. The continuous-data output $C(s)$ can be expressed in terms of input as.

$$C(s) = G(s)R(s) - \frac{G(s)H(s)}{1 + GH^*(s)}GR^*(s)$$

Characteristics Equation

Characteristics equation plays an important role in the study of linear systems. As said earlier, an n^{th}order LTI discrete data system can be represented by an n^{th} order difference equation,

$$c(k+n) + a_{n-1}c(k+n-1) + a_{n-2}c(k+n-2) + ... + a_1c(k+1) + a_0c(k)$$

$$= b_m r(k+m) + b_{m-1}r(k+m-1) + ... + b_0 r(k)$$

where *r(k)* and *c(k)* denote input and output sequences respectively

The input output relation can be obtained by taking Z-transformation on both sides, with zero initial conditions, as

$$G(z) = \frac{C(z)}{R(z)}$$

$$= \frac{b_m z^m + b_{m-1}z^{m-1} + ... + b_0}{z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0} \qquad (6)$$

The characteristics equation is obtained by equating the denominator of *G(z)* to 0, as

$$z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0$$

Example

Consider the forward path transfer function as $G(s) = \frac{2}{s(s+2)}$ and the feedback transfer function as 1. If the sampler is placed in the forward path, find out the characteristics equation of the overall system for a sampling period T = 0.1 sec.

Solution:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

Since the feedback transfer function is 1,

$$G(z) = GH(z) = _z\left[\frac{2}{s(s+2)}\right]$$

$$= \frac{2}{2}\frac{(1-e^{-2T})z}{(z-1)(z-e^{-2T})}$$

$$= \frac{0.18z}{z^2-1.82z+0.82}$$

$$\Rightarrow \frac{C(z)}{R(z)} = \frac{0.18z}{z^2-1.64z+0.82}$$

So, the characteristics equation of the system is $z^2 - 1.64z + 0.82 = 0$.

Causality and Physical Realizability

In a causal system, the output does not precede the input. In other words, in a causal system, the output depends only on the past and presents inputs, not on the future ones.

The transfer function of a causal system is physically realizable, i.e., the system can be realized by using physical elements.

For a causal discrete data system, the power series expansion of its transfer function must not contain any positive power in *z* Positive power in *z* indicates prediction. Therefore, in the transfer function (6), *n* must be greater than or equal to *m*.

$m = n \Rightarrow$ **proper transfer function**
$m < n \Rightarrow$ **strictly proper Transfer function**

Unit – III
State Space analysis and concepts of controllability and observability

State space analysis and the concepts of Controllability and observability
State Space Representation of discrete time systems – State transition matrix and methods of evaluation – Discretization of continuous – Time state equations – Concepts of controllability and observability – Tests (without proof).

Unit Objectives:

After reading this Unit, you should be able to understand:

- To represent the discrete–time systems in state–space model and evaluation of state transition matrix.

Unit Outcomes:

- Finally, the conventional and state–space methods of design are also introduced.

- **INTRODUCTION**

The analysis and design of control system are carried out using transfer functions together with a variety of graphical techniques such as root locus plots and Nyquist plots based on the input-output relations of the system. They are applicable only to linear time invariant systems having a single input and single output (SISO). Hence a new approach to control system analysis and design is evolved, which can be applied to the design of optimal and adaptive control system, which are mostly time varying and/or non-linear multiple inputs and multiple outputs(MIMO). This new approach is based on the concept of state, which includes the initial conditions in the design.

- **Advantages of state-space technique:**
  - It is possible to analyses time-varying or time-invarying linear or non-linear, single or multiple input-output systems.
  - State equations are highly compatible for simulation on analog or digital computer.
  - It is possible to optimize the system useful for optimal design.
  - State space analysis gives us the information about the internal behavior of the system,

as well as the input and output behavior.

- State-space techniques can be used to find the stability of a negative feed-back system, when the feed-back signal is other than the output signal.
- It is possible to include initial conditions in state space technique.

- **CONCEPT OF STATE, STATE VARIABLES & STATE VECTOR**

State: The state of a dynamic system is the smallest set of variables, called state variables such that the knowledge of these variables at $t = t_o$ together with the input for $t>t_o$. Completely determine the behaviour of the system for anytime $t> t_o$. Note that in dealing with linear time invariant systems, we usually choose the reference time to be zero.

State Variables: The state variables of a dynamic system are the smallest set of variables which determine the state of the dynamic system. If at least n variables $x_1(k)$, $x_2(k)$,----------- $x_n(k)$ are needed to completely describe the behavior of a dynamic system, then such n variables $x_1(k)$, $x_2(k)$,-----------$x_n(k)$ are called a set of state variables

The generic structure of a state-space model of a n^{th} order continuous time dynamical system with *m* input and *p* output is given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad \text{: State Equation} \\ y(t) &= Cx(t) + Du(t) \quad \text{: Output Equation} \end{aligned} \tag{1}$$

where, $x(t)$ is the n dimensional state vector, $u(t)$ is the m dimensional input vector, $y(t)$ is the $p$dimensional output vector and $A \varepsilon R^{n \times n}$, $B \varepsilon R^{n \times m}$, , $C \varepsilon R^{p \times n}$, $D \varepsilon R^{p \times m}$,

Example

Consider a nth order differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = u$$

Define following variables,

$$
\begin{aligned}
y &= x_1 \\
\frac{dy}{dt} &= x_2 \\
\vdots &= \vdots \\
\frac{d^{n-1}y}{dt^{n-1}} &= x_n \\
\frac{d^n y}{dt^n} &= -a_1 x_{n-1} - a_2 x_{n-2} - \ldots - a_n x_1 + u
\end{aligned}
$$

The nth order differential equation may be written in the form of *n* first order differential equations as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots &= \vdots \\
\dot{x}_n &= -a_1 x_{n-1} - a_2 x_{n-2} - \ldots - a_n x_1 + u
\end{aligned}
$$

or in matrix form as,

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

where

$$
A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
$$

The output can be one of states or a combination of many states. Since, $y = x_1$,

$$y = [1\ 0\ 0\ 0\ \ldots\ 0]\boldsymbol{x}$$

➢ **Correlation between state variable and transfer functions models**

The transfer function corresponding to state variable model (1) , when *u* and *y* are scalars, is:

$$\begin{aligned} G(s) &= \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \\ &= \frac{Q(s)}{|sI - A|} \end{aligned} \qquad (2)$$

where $|sI - A|$ is the characteristic polynomial of the system.

- **Solution of Continuous Time State Equation**

The solution of state equation (1) is given as

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

where $e^{At} = \Phi(t)$ is known as the state transition matrix and $x(t_0)$ is the initial state of the system.

- **State Variable Analysis of Digital Control Systems**

The discrete time systems, as discussed earlier, can be classified in two types.

1. Systems that result from sampling the continuous time system output at discrete instants only, i.e., sampled data systems.

2. Systems which are inherently discrete where the system states are defined only at discrete time instants and what happens in between is of no concern to us.

- **State Equations of Sampled Data Systems**

Let us assume that the following continuous time system is subject to sampling process with an interval of T.

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + Bu(t) \quad &: \text{State Equation} \\ y(t) &= C\boldsymbol{x}(t) + Du(t) \quad &: \text{Output Equation} \end{aligned}$$

We know that the solution to above state equation is:

$$\boldsymbol{x}(t) = \Phi(t - t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \Phi(t - \tau)Bu(\tau)d\tau$$

Since the inputs are constants in between two sampling instants, one can write:

$$u(\tau) = u(kT) \quad \text{for} \quad kT \leq \tau \leq (k+1)T$$

which implies that the following expression is valid within the interval $kT \leq \tau \leq (k+1)T$ if we consider $t_0 = kT$:

$$\boldsymbol{x}(t) = \Phi(t - kT)\boldsymbol{x}(kT) + \int_{kT}^{t} \Phi(t-\tau)Bu(kT)d\tau$$

Let us denote $\int_{kT}^{t} \Phi(t-\tau)Bd\tau$ by $\theta(t - KT)$. Then we can write:

$$\boldsymbol{x}(t) = \Phi(t - kT)\boldsymbol{x}(kT) + \theta(t - KT)u(kT)$$

If $t = (k+1)T$,

$$\boldsymbol{x}((k+1)T) = \Phi(T)\boldsymbol{x}(kT) + \theta(T)u(kT) \tag{4}$$

where $\Phi(T) = e^{AT}$ and $\theta(T) = \int_{kT}^{(k+1)T} \Phi((k+1)T - \tau)Bd\tau$.

If $t' = \tau - kT$, we can rewrite $\theta(T)$ as $\theta(T) = \int_{0}^{T} \Phi(T - t')Bdt'$.

Equation (4) has a similar form as that of equation (3) if we consider $\phi(T) = \bar{A}$ and $\theta(T) = \bar{B}$. Similarly by setting $t = kT$, one can show that the output equation also has a similar form as that of the continuous time one.

When $T = 1$,

$$\begin{aligned} \boldsymbol{x}(k+1) &= \Phi(1)\boldsymbol{x}(k) + \theta(1)u(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

State Equations of Inherently Discrete Systems

When a discrete system is composed of all digital signals, the state and output equations can be described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

Discrete Time Approximation of A Continuous Time State Space Model

Let us consider the dynamical system described by the state space model (3). By approximating the derivative at $t = kT$ using forward difference, we can write:

$$\begin{aligned} \dot{\boldsymbol{x}}(t)\big|_{t=kT} &= \frac{1}{T}[\boldsymbol{x}((k+1)T) - \boldsymbol{x}(kT)] \\ \Rightarrow \frac{1}{T}[\boldsymbol{x}((k+1)T) - \boldsymbol{x}(kT)] &= A\boldsymbol{x}(kT) + Bu(kT) \\ \text{and,}\ \ y(kT) &= C\boldsymbol{x}(kT) + Du(kT) \end{aligned}$$

Rearranging the above equations,

$$\begin{aligned} \boldsymbol{x}((k+1)T) &= (I + TA)\boldsymbol{x}(kT) + TBu(kT) \\ \text{If, } T = 1 \Rightarrow \boldsymbol{x}(k+1) &= (I + A)\boldsymbol{x}(k) + Bu(k) \\ \text{and } y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

We can thus conclude from the discussions so far that the discrete time state variable model of a system can be described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

where A, B are either the descriptions of an all digital system or obtained by sampling the continuous time process.

State Space Model to Transfer Function

Consider a discrete state variable model

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned} \quad (1)$$

Taking the Z-transform on both sides of Eqn. (1), we get

$$\begin{aligned} zX(z) - z\boldsymbol{x}_0 &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

where $\boldsymbol{x}_0$ is the initial state of the system.

$$\begin{aligned} &\Rightarrow \qquad (zI - A)X(z) = z\boldsymbol{x}_0 + BU(z) \\ &\text{or,} \quad X(z) = (zI - A)^{-1}z\boldsymbol{x}_0 + (zI - A)^{-1}BU(z) \end{aligned}$$

To find out the transfer function, we assume that the initial conditions are zero, i.e., $x_0 = 0$, thus

$$Y(z) = \Big(C(zI - A)^{-1}B + D\Big)U(z)$$

Therefore, the transfer function becomes

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D \quad (2)$$

which has the same form as that of a continuous time system.

- **Various Canonical Forms**

We have seen that transform domain analysis of a digital control system yields a transfer function of the following form.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\beta_0 z^m + \beta_1 z^{m-1} + \ldots + \beta_m}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n} \quad m \leq n \quad (3)$$

Various canonical state variable models can be derived from the above transfer function model.

- **Controllable canonical form**

Consider the transfer function as given in Eqn. (3). Without loss of generality, we assume $m = n$. Let

$$\frac{\bar{X}(z)}{U(z)} = \frac{1}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n}$$

In time domain, the above equation may be written as

$$\bar{x}(k+n) + \alpha_1 \bar{x}(k+n-1) + \ldots + \alpha_n \bar{x}(k) = u(k)$$

Now, the output $Y(z)$ may be written in terms of $\bar{X}(z)$ as

$$Y(z) = \left(\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_n\right) \bar{X}(z)$$

or in time domain as

$$y(k) = \beta_0 \bar{x}(k+n) + \beta_1 \bar{x}(k+n-1) + \ldots + \beta_n \bar{x}(k)$$

The block diagram representation of above equations is shown in Figure 1. State variables are selected as shown in Figure 1.

The state equations are then written as:

$$\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= x_3(k) \\
\vdots &= \vdots \\
x_n(k+1) &= -\alpha_n x_1(k) - \alpha_{n-1} x_2(k) - \ldots - \alpha_1 x_n(k) + u(k)
\end{aligned}$$

Output equation can be written as by following the Figure 1.

$$y(k) = (\beta_n - \alpha_n\beta_0)x_1(k) + (\beta_{n-1} - \alpha_{n-1}\beta_0)x_2(k) + \ldots + (\beta_1 - \alpha_1\beta_0)x_n(k) + \beta_0 u(k)$$

In state space form, we have

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) + Du(k) \end{aligned} \qquad (4)$$

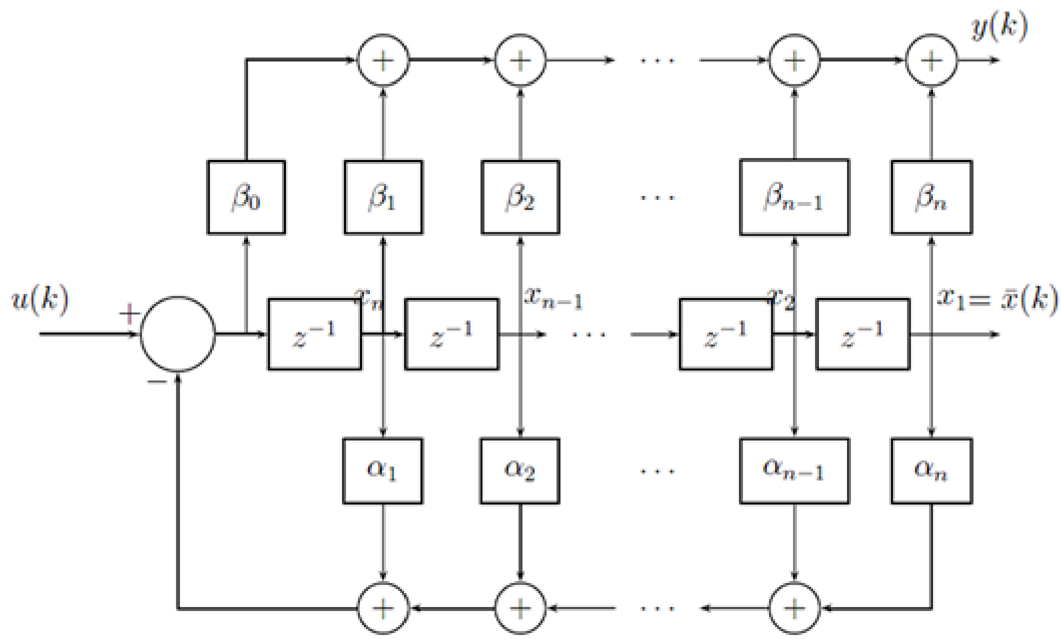


Figure 1: Block Diagram representation of controllable canonical form

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \ldots & -\alpha_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\beta_n - \alpha_n\beta_0 \;\; \beta_{n-1} - \alpha_{n-1}\beta_0 \;\; \ldots \;\; \beta_1 - \alpha_1\beta_0] \qquad D = \beta_0$$

- **Observable Canonical Form**

Equation (3) may be rewritten as

$$\left(z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n\right) Y(z) = \left(\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_n\right) U(z)$$

$$\text{or, } z^n[Y(z) - \beta_0 U(z)] + z^{n-1}[\alpha_1 Y(z) - \beta_1 U(z)] + \ldots + [\alpha_n Y(z) - \beta_n U(z)] = 0$$

$$Y(z) = \beta_0 U(z) - z^{-1}[\alpha_1 Y(z) - \beta_1 U(z)] - \ldots - z^{-n}[\alpha_n Y(z) - \beta_n U(z)]$$

The corresponding block diagram is shown in Figure 2.

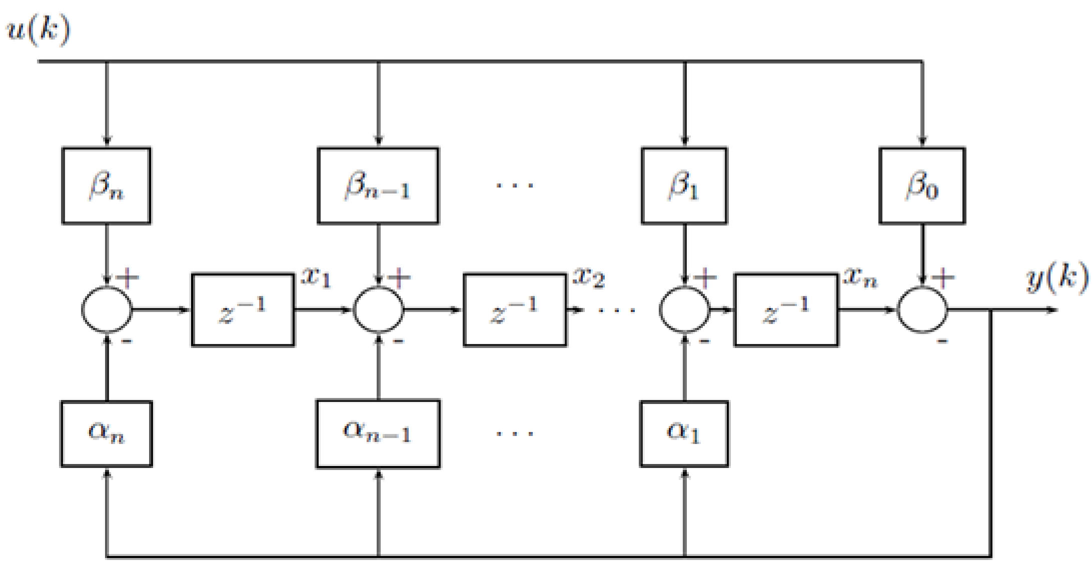


Figure 2: Block Diagram representation of observable canonical form

Choosing the outputs of the delay blocks as the state variables, we have following state equations

$$\begin{aligned}
x_n(k+1) &= x_{n-1}(k) - \alpha_1(x_n(k) + \beta_0 u(k)) + \beta_1 u(k) \\
x_{n-1}(k+1) &= x_{n-2}(k) - \alpha_2(x_n(k) + \beta_0 u(k)) + \beta_2 u(k) \\
\vdots &= \vdots \\
x_1(k+1) &= -\alpha_n(x_n(k) + \beta_0 u(k)) + \beta_n u(k) \\
y(k) &= x_n(k) + \beta_0 u(k)
\end{aligned}$$

This can be rewritten in matrix form (4) with

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -\alpha_n \\ 1 & 0 & 0 & \dots & 0 & -\alpha_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & -\alpha_1 \end{bmatrix} \quad B = \begin{bmatrix} \beta_n - \alpha_n\beta_0 \\ \beta_{n-1} - \alpha_{n-1}\beta_0 \\ \vdots \\ \beta_1 - \alpha_1\beta_0 \end{bmatrix} \quad C = [0 \;\; 0 \;\; \dots \;\; 1] \quad D = \beta_0$$

- **Duality**

In previous two sections we observed that the system matrix *A* in observable canonical form is transpose of the system matrix in controllable canonical form. Similarly, control matrix *B* in observable canonical form is transpose of output matrix *C* in controllable canonical form. So also output matrix *C* in observable canonical form is transpose of control matrix *B* in controllable canonical form.

- **Jordan Canonical Form**

In Jordan canonical form, the system matrix *A* represents a diagonal matrix for distinct poles which basically form the diagonal elements of *A*.

Assume that $z = \lambda_i, \;\; i = 1, 2, \dots, n$ are the distinct poles of the given transfer function (3). Then partial fraction expansion of the transfer function yields

$$\begin{aligned} \frac{Y(z)}{U(z)} &= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \dots + \bar{\beta}_n}{z^n + \alpha_1 z^{n-1} + \dots + \alpha_n} \\ &= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \dots + \bar{\beta}_n}{(z-\lambda_1)(z-\lambda_2)\dots(z-\lambda_n)} \\ &= \beta_0 + \frac{r_1}{z-\lambda_1} + \frac{r_2}{z-\lambda_2} + \dots + \frac{r_n}{z-\lambda_n} \end{aligned} \qquad (5)$$

A parallel realization of the transfer function (5) is shown in Figure 3.

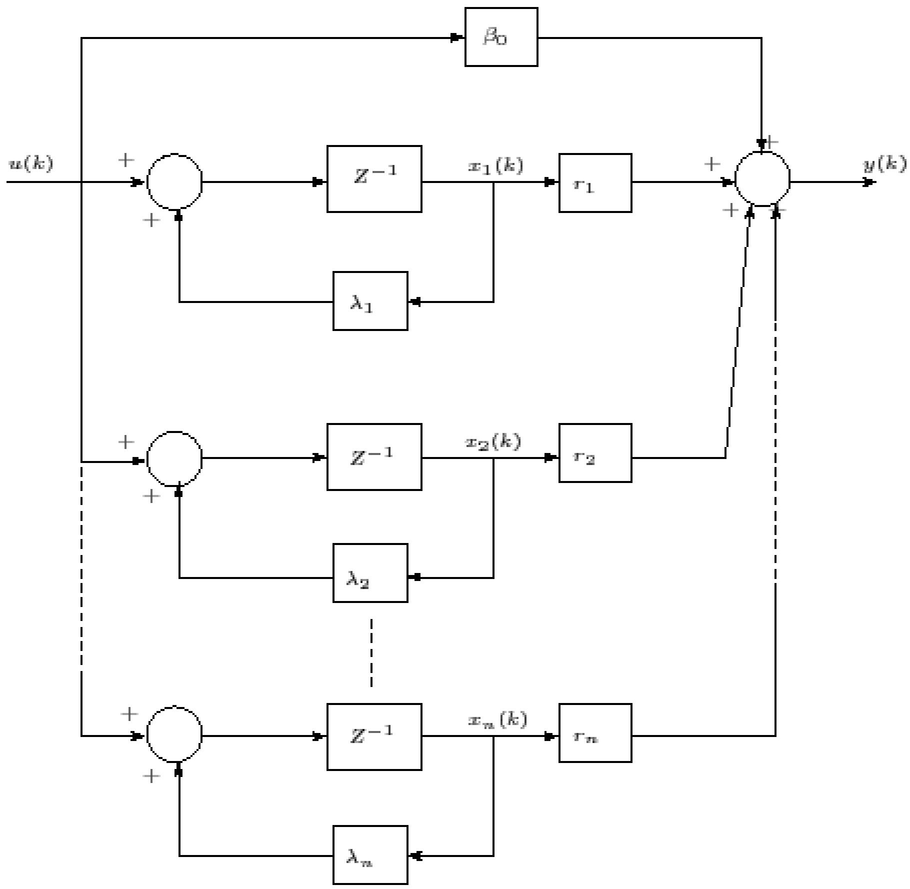


Figure 3: Block Diagram representation of Jordan canonical form

Considering the outputs of the delay blocks as the state variables, we can construct the state model in matrix form (4), with

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad C = [r_1 \ \ r_2 \ \ \dots \ \ r_n] \quad D = \beta_0$$

When the matrix A has repeated eigenvalues, it cannot be expressed in a proper diagonal form. However, it can be expressed in a Jordan canonical form which is nearly a diagonal matrix. Let us consider that the system has eigenvalues, λ_1, λ_1, λ_2 and λ_3. In that case, A matrix in Jordan canonical form will be

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

1. The diagonal elements of the matrix A are eigenvalues of the same.
2. The elements below the principal diagonal are zero.
3. Some of the elements just above the principal diagonal are one.
4. The matrix can be divided into a number of blocks, called Jordan blocks, along the diagonal. Each block depends on the multiplicity of the eigenvalue associated with it. For example, Jordan block associated with a eigenvalue z_1 of multiplicity 4 can be written as

$$A = \begin{bmatrix} z_1 & 1 & 0 & 0 \\ 0 & z_1 & 1 & 0 \\ 0 & 0 & z_1 & 1 \\ 0 & 0 & 0 & z_1 \end{bmatrix}$$

Example: Consider the following discrete transfer function.

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24}$$

Find out the state variable model in 3 different canonical forms.

Solution:

The state variable model in controllable canonical form can directly be derived from the transfer function, where the A, B, C and D matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -0.24 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.04 & 0.17 \end{bmatrix}, \quad D = 0$$

The matrices in state model corresponding to observable canonical form are obtained as,

$$A = \begin{bmatrix} 0 & -0.24 \\ 1 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.04 \\ 0.17 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

To find out the state model in Jordan canonical form, we need to fact expand the transfer function using partial fraction, as

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24} = \frac{0.352}{z - 0.8} + \frac{-0.182}{z - 0.3}$$

Thus the A, B, C and D matrices will be:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.352 & -0.182 \end{bmatrix}, \quad D = 0$$

Characteristic Equation, eigenvalues and eigen vectors

For a discrete state space model, the characteristic equation is defined as

$$|zI - A| = 0$$

The roots of the characteristic equation are the eigenvalues of matrix *A*.

1. If $det(A) \neq 0$, i.e., *A* is nonsingular and λ_1, λ_2, $\cdots$, λ_n are the eigenvalues of *A*, then, $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, $\cdots$, $\frac{1}{\lambda_n}$ will be the eigenvalues of A^{-1}.

2. Eigenvalues of *A* and A^T are same when *A* is a real matrix.

3. If *A* is a real symmetric matrix then all its eigenvalues are real.

The $n \times 1$ vector v_i which satisfies the matrix equation

$$A v_i = \lambda_i v_i \qquad (1)$$

where $\lambda_i, \;\; i = 1, 2, \cdots, n$ denotes the i^{th} eigenvalue, is called the eigen vector of *A* associated with the eigenvalue λ_i. If eigenvalues are distinct, they can be solved directly from equation (1).

Properties of eigen vectors

1. An eigen vector cannot be a null vector.

2. If v_i is an eigen vector of *A* then mv_i is also an eigen vector of *A* where *m* is a scalar.

3. If *A* has *n* distinct eigenvalues, then the *n* eigen vectors are linearly independent.

- **Eigen vectors of multiple order eigenvalues**

When the matrix A an eigenvalue λ of multiplicity m, a full set of linearly independent may not exist. The number of linearly independent eigen vectors is equal to the degeneracy d of $\lambda I - A$. The degeneracy is defined as

$$d = n - r$$

where n is the dimension of A and r is the rank of $\lambda I - A$. Furthermore,

$$1 \leq d \leq m$$

- **Similarity Transformation and Diagonalization**

Square matrices *A* and $\bar{A}$ are similar if

$$\begin{aligned} AP &= P\bar{A} \\ \text{or,} \quad \bar{A} &= P^{-1}AP \\ \text{and,} \quad A &= P\bar{A}P^{-1} \end{aligned}$$

The non-singular matrix *P* is called similarity transformation matrix. It should be noted that eigenvalues of a square matrix *A* are not altered by similarity transformation.

- **Diagonalization:**

If the system matrix *A* of a state variable model is diagonal then the state dynamics are decoupled from each other and solving the state equations become much more simpler.

In general, if A has distinct eigenvalues, it can be diagonalized using similarity transformation. Consider a square matrix A which has distinct eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ \lambda_n$. It is required to find a transformation matrix P which will convert A into a diagonal form

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

through similarity transformation AP = P∧ . If $v_1,\ v_2,\ \ldots,\ v_n$ are the eigenvectors of matrix Acorresponding to eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ \lambda_n$, then we know $Av_i = \lambda_i v_i$. This gives

$$A\,[v_1\ v_2\ \ldots\ v_n] = [v_1\ v_2\ \ldots\ v_n]\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Thus $P = [v_1\ v_2\ \ldots\ v_n]$. Consider the following state model.

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

If P transforms the state vector $\boldsymbol{x}(k)$ to $\boldsymbol{z}(k)$ through the relation

$\boldsymbol{x}(k) = P\boldsymbol{z}(k),$ or, $\boldsymbol{z}(k) = P^{-1}\boldsymbol{x}(k)$

then the modified state space model becomes

$$\boldsymbol{z}(k+1) = P^{-1}AP\boldsymbol{z}(k) + P^{-1}Bu(k)$$

where $P^{-1}AP = \Lambda$.

- **Computation of $\Phi(t)$**

We have seen that to derive the state space model of a sampled data system, we need to know the continuous time state transition matrix $\Phi(t) = e^{At}$.

- **Using Inverse Laplace Transform**

For the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$, the state transition matrix e^{At} can be computed as,

$$e^{At} = \mathscr{L}^{-1}\{(sI - A)^{-1}\}$$

- **Using Similarity Transformation**

If Λ is the diagonal representation of the matrix A, then $\Lambda = P^{-1}AP$. When a matrix is in diagonal form, computation of state transition matrix is straight forward:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

Given $e^{\Lambda t}$, we can show that

$$e^{At} = Pe^{\Lambda t}P^{-1}$$

Proof.

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \dots \\ \Rightarrow P^{-1}e^{At}P &= P^{-1}\left[I + At + \frac{1}{2!}A^2t^2 + \dots\right]P \\ &= I + P^{-1}APt + \frac{1}{2!}P^{-1}APP^{-1}APt^2 + \dots \\ &= I + \Lambda t + \frac{1}{2!}\Lambda^2t^2 + \dots \\ &= e^{\Lambda t} \\ \Rightarrow e^{At} &= Pe^{\Lambda t}P^{-1} \end{aligned}$$

- **Using Caley Hamilton Theorem**

Every square matrix A satisfies its own characteristic equation. If the characteristic equation is

$$\triangle(\lambda) = |\lambda I - A| = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n = 0$$

then,

$$\triangle(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0$$

Application: Evaluation of any function $f(\lambda)$ and $f(A)$

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n + \cdots$$ order ∞

$$\frac{f(\lambda)}{\triangle(\lambda)} = q(\lambda) + \frac{g(\lambda)}{\triangle(\lambda)}$$

$$\begin{aligned} f(\lambda) &= q(\lambda)\triangle(\lambda) + g(\lambda) \\ &= g(\lambda) \\ &= \beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1} \quad \text{order } n-1 \end{aligned}$$

If A has distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, then,

$$f(\lambda_i) = g(\lambda_i), \quad i = 1, \cdots, n$$

The solution will give rise to $\beta_0, \beta_1, \cdots, \beta_{n-1}$, then

$$f(A) = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$$

If there are multiple roots (multiplicity = 2), then

$$f(\lambda_i) = g(\lambda_i) \qquad (2)$$

$$\frac{\partial}{\partial \lambda_i} f(\lambda_i) = \frac{\partial}{\partial \lambda_i} g(\lambda_i) \qquad (3)$$

Example 1:

If $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

then compute the state transition matrix using Caley Hamilton Theorem.

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)^2(\lambda-2) = 0 \Rightarrow \lambda_1$$

(with multiplicity 2) , $\lambda_2 = 2$

Let $f(\lambda) = e^{\lambda t}$ and $g(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$

Then using (2) and (3), we can write

$$\begin{aligned} f(\lambda_1) &= g(\lambda_1) \\ \frac{\partial}{\partial\lambda_1} f(\lambda_1) &= \frac{\partial}{\partial\lambda_1} g(\lambda_1) \\ f(\lambda_2) &= g(\lambda_2) \end{aligned}$$

This implies

$$\begin{aligned} e^t &= \beta_0 + \beta_1 + \beta_2 \quad (\lambda_1 = 1) \\ te^t &= \beta_1 + 2\beta_2 \quad (\lambda_1 = 1) \\ e^{2t} &= \beta_0 + 2\beta_1 + 4\beta_2 \quad (\lambda_2 = 2) \end{aligned}$$

Solving the above equations

$$\beta_0 = e^{2t} - 2te^t, \qquad \beta_1 = 3te^t + 2e^t - 2e^{2t}, \qquad \beta_2 = e^{2t} - e^t - te^t$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A + \beta_2 A^2 \\ &= \begin{bmatrix} 2e^t - e^{2t} & 0 & 2e^t - 2e^{2t} \\ 0 & e^t & 0 \\ e^{2t} - e^t & 0 & 2e^{2t} - e^t \end{bmatrix} \end{aligned}$$

Example 2 For the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$, where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Compute e^{At} using 3 different techniques.

Solution: Eigenvalues of matrix A are $1 \pm j1$.

Method 1:

$$\begin{aligned} e^{At} &= \mathscr{L}^{-1}(sI-A)^{-1} = \mathscr{L}^{-1} = \begin{bmatrix} s-1 & -1 \\ 1 & s-1 \end{bmatrix}^{-1} \\ &= \mathscr{L}^{-1}\frac{1}{s^2-2s+2}\begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix} \\ &= \mathscr{L}^{-1}\begin{bmatrix} \frac{s-1}{(s-1)^2+1} & \frac{1}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-1}{(s-1)^2+1} \end{bmatrix} \\ &= \begin{bmatrix} e^t \cos\ t & e^t \sin\ t \\ -e^t \sin\ t & e^t \cos\ t \end{bmatrix} \end{aligned}$$

Method 2:

$e^{At} = Pe^{\Lambda t}P^{-1}$ where $e^{\Lambda t} = \begin{bmatrix} e^{(1+j)t} & 0 \\ 0 & e^{(1-j)t} \end{bmatrix}$. Eigen values are $1 \pm j$. The corresponding eigenvectors are found by using equation $Av_i = \lambda_i v_i$ as follows:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (1+j)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Taking $v_1 = 1$, we get $v_2 = j$. So, the eigenvector corresponding to $1+j$ is $\begin{bmatrix} 1 \\ j \end{bmatrix}$ and the one corresponding to $1-j$ is $\begin{bmatrix} 1 \\ -j \end{bmatrix}$. The transformation matrix is given by

$$P = [v_1 \ \ v_2] = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

Now,

$$
\begin{aligned}
e^{At} &= Pe^{\Lambda t}P^{-1} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}\begin{bmatrix} e^{(1+j)t} & 0 \\ 0 & e^{(1-j)t} \end{bmatrix}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} e^{(1+j)t} & e^{(1-j)t} \\ je^{(1+j)t} & -je^{(1-j)t} \end{bmatrix}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} e^{(1+j)t} + e^{(1-j)t} & -j\left(e^{(1+j)t} - e^{(1-j)t}\right) \\ j\left(e^{(1+j)t} - e^{(1-j)t}\right) & e^{(1+j)t} + e^{(1-j)t} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2e^{t}\cos\ t & -j(j)e^{t}2\sin\ t \\ e^{t}(j)(j)2\sin\ t & 2e^{t}\cos\ t \end{bmatrix} \\
&= \begin{bmatrix} e^{t}\cos\ t & e^{t}\sin\ t \\ -e^{t}\sin\ t & e^{t}\cos\ t \end{bmatrix}
\end{aligned}
$$

Method 3:

Caley Hamilton Theorem

The eigenvalues are .

$$
\begin{aligned}
e^{\lambda_1 t} &= \beta_0 + \beta_1\lambda_1 \\
e^{\lambda_2 t} &= \beta_0 + \beta_1\lambda_2
\end{aligned}
$$

Solving,

$$
\begin{aligned}
\beta_0 &= \frac{1}{2}(1+j)e^{(1+j)t} + \frac{1}{2}(1-j)e^{(1-j)t} \\
\beta_1 &= \frac{1}{2j}\left(e^{(1+j)^t} - e^{(1-j)^t}\right)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
e^{At} &= \beta_0 I + \beta_1 A \\
&= \begin{bmatrix} e^{t}\cos\ t & e^{t}\sin t \\ -e^{t}\sin t & e^{t}\cos\ t \end{bmatrix}
\end{aligned}
$$

We will now show through an example how to derive discrete state equation from a continuous one.

Example: Consider the following state model of a continuous time system.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= x_1(t) \end{aligned}$$

If the system is under a sampling process with period T, derive the discrete state model of the system.

To derive the discrete state space model, let us first compute the state transition matrix of the continuous time system using Caley Hamilton Theorem.

$$\triangle(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$$

$$\text{Let } f(\lambda) = e^{\lambda t}$$

This implies

$$\begin{aligned} e^{t} &= \beta_0 + \beta_1 \quad (\lambda_1 = 1) \\ e^{2t} &= \beta_0 + 2\beta_1 \quad (\lambda_2 = 2) \end{aligned}$$

Solving the above equations

$$\beta_1 = e^{2t} - e^{t} \qquad \beta_0 = 2e^{t} - e^{2t}$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^{t} & e^{2t} - e^{t} \\ 0 & e^{2t} \end{bmatrix} \end{aligned}$$

Thus the discrete state matrix A is given as

$$A = \Phi(T) = \begin{bmatrix} e^{T} & e^{2T} - e^{T} \\ 0 & e^{2T} \end{bmatrix}$$

The discrete input matrix B can be computed as

$$\begin{aligned} B &= \Theta(T) = \int_0^T \Phi(T-t') \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \int_0^T \begin{bmatrix} e^T.e^{-t'} & e^{2T}.e^{-2t'} - e^T.e^{-t'} \\ 0 & e^{2T}.e^{-2t'} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \begin{bmatrix} e^T - 1 & 0.5e^{2T} - e^T + 0.5 \\ 0 & 0.5e^{2T} - 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5e^{2T} - e^T + 0.5 \\ 0.5e^{2T} - 0.5 \end{bmatrix} \end{aligned}$$

The discrete state equation is thus described by

$$\begin{aligned} \mathbf{x}((k+1)T) &= \begin{bmatrix} e^T & e^{2T} - e^T \\ 0 & e^{2T} \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} 0.5e^{2T} - e^T + 0.5 \\ 0.5e^{2T} - 0.5 \end{bmatrix} u(kT) \\ y(kT) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(kT) \end{aligned}$$

When T = 1 , the state equations become

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 2.72 & 4.67 \\ 0 & 7.39 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1.48 \\ 3.19 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$

Solution to Discrete State Equation

In this lecture we would discuss about the solution of discrete state equation, computation of discrete state transition matrix and state diagram.

Consider the following state model of a discrete time system:

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

where the initial conditions are $x(0)$ and $u(0)$. Putting $k = 0$ in the above equation, we get

$$\boldsymbol{x}(1) = A\boldsymbol{x}(0) + Bu(0)$$

Similarly if we put $k = 1$, we would get

$$\begin{aligned} \boldsymbol{x}(2) &= A\boldsymbol{x}(1) + Bu(1) \\ \text{Putting the expression of } \boldsymbol{x}(1) \Rightarrow \boldsymbol{x}(2) &= A^2\boldsymbol{x}(0) + ABu(0) + Bu(1) \end{aligned}$$

For $k = 2$,

$$\begin{aligned} \boldsymbol{x}(3) &= A\boldsymbol{x}(2) + Bu(2) \\ &= A^3\boldsymbol{x}(0) + A^2Bu(0) + ABu(1) + Bu(2) \end{aligned}$$

and so on. If we combine all these equations, we would get the following expression as a general solution:

$$\boldsymbol{x}(k) = A^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i)$$

As seen in the above expression, $x(k)$ has two parts. One is the contribution due to the initial state $x(0)$ and the other one is the contribution of the external input $u(i)$ for $i = 0, 1, 2, \cdots, k-1$. When the input is zero, solution of the homogeneous state equation $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k)$ can be written as

$$\boldsymbol{x}(k) = A^k\boldsymbol{x}(0)$$

where $A^k = \phi(k)$ is the state transition matrix.

Evaluation of $\phi(k)$

Similar to the continuous time systems, the state transition matrix of a discrete state model can be evaluated using the following different techniques.

- **Using Inverse Z-transform**:

$$\phi(k) = \mathscr{Z}^{-1}\{(zI - A)^{-1}\}$$

- **Using Similarity Transformation** If ∧ is the diagonal representation of the matrix A, then $\Lambda = P^{-1}AP$. When a matrix is in diagonal form, computation of state transition matrix is straight forward:

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k t & \dots & 0 \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

Given Λ^k, we can compute $A^k = P\Lambda^k P^{-1}$

- **Using Caley Hamilton Theorem**

Example Compute A^k for the following system using three different techniques and hence find $y(k)$for $k \geq 0$.

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-1)^k; \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y(k) &= x_2(k) \end{aligned}$$

Solution: $A = \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix}$ and eigenvalues of A are - 0.3 and - 0.7

Method 1 :

$$A^k = \mathscr{Z}^{-1}(zI - A)^{-1} = \mathscr{Z}^{-1}\left\{ \begin{bmatrix} z-1 & -1 \\ 1 & z-1 \end{bmatrix}^{-1} \right\}$$

$$A^k = \mathscr{Z}^{-1}\begin{bmatrix} \frac{z+1}{z^2+z+0.21} & \frac{1}{z^2+z+0.21} \\ \frac{-0.21}{z^2+z+0.21} & \frac{z}{z^2+z+0.21} \end{bmatrix}$$

$$= \mathscr{Z}^{-1}\begin{bmatrix} \frac{1.75}{z+0.3} - \frac{0.75}{z+0.7} & \frac{2.5}{z+0.3} - \frac{2.5}{z+0.7} \\ \frac{-0.525}{z+0.3} + \frac{0.525}{z+0.7} & \frac{-0.75}{z+0.3} + \frac{1.75}{z+0.7} \end{bmatrix}$$

$$= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}$$

Method 2:

$A^k = P\Lambda^k P^{-1}$ where $\Lambda^k = \begin{bmatrix} (-0.3)^k & 0 \\ 0 & (-0.7)^k \end{bmatrix}$.

Eigen values are - 0.3 and - 0.7. The corresponding eigenvectors are found, by using equation $Av_i = \lambda_i v_i$, as $\begin{bmatrix} 1 \\ -0.3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -0.7 \end{bmatrix}$ respectively. The transformation matrix is given by

$$P = \begin{bmatrix} 1 & 1 \\ -0.3 & -0.7 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1.75 & 2.5 \\ -0.75 & -2.5 \end{bmatrix}$$

Thus,

$$A^k = P\Lambda^k P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -0.3 & -0.7 \end{bmatrix}\begin{bmatrix} (-0.3)^k & 0 \\ 0 & (-0.7)^k \end{bmatrix}\begin{bmatrix} 1.75 & 2.5 \\ -0.75 & -2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}$$

Method 3:

Caley Hamilton Theorem

The eigenvalues are - 0.3 and - 0.7.

$$
\begin{aligned}
(-0.3)^k &= \beta_0 - 0.3\beta_1 \\
(-0.7)^k &= \beta_0 - 0.7\beta_1
\end{aligned}
$$

Solving,

$$
\begin{aligned}
\beta_0 &= 1.75(-0.3)^k - 0.75(-0.7)^k \\
\beta_1 &= 2.5(-0.3)^k - 2.5(-0.7)^k
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\phi(k) = A^k &= \beta_0 I + \beta_1 A \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}
\end{aligned}
$$

The solution $x(k)$ is

$$
\begin{aligned}
\boldsymbol{x}(k) &= A^k \boldsymbol{x}(0) + \sum_{i=0}^{k-1} A^{k-1-i} B u(i) \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k \end{bmatrix} + \sum_{i=0}^{k-1} \begin{bmatrix} 2.5(-0.3)^{k-1-i} - 2.5(-0.7)^{k-1-i} \\ -0.75(-0.3)^{k-1-i} + 1.75(-0.7)^{k-1-i} \end{bmatrix} (-1)^i
\end{aligned}
$$

Since $y(k) = x_2(k)$, we can write

$$
\begin{aligned}
y(k) &= -0.525(-0.3)^k + 0.525(-0.7)^k + \sum_{i=0}^{k-1} [-0.75(-0.3)^{k-1-i} + 1.75(-0.7)^{k-1-i}](-1)^i \\
&= -0.525(-0.3)^k + 0.525(-0.7)^k - 0.75(-0.3)^{k-1} \sum_{i=0}^{k-1} (1/0.3)^i + 1.75(-0.7)^{k-1} \sum_{i=0}^{k-1} (1/0.7)^i
\end{aligned}
$$

Now,

$$\sum_{i=0}^{k-1}(1/0.3)^i = \sum_{i=0}^{k-1}(3.33)^i = \frac{1-(3.33)^k}{1-3.33} = -0.43[1-(3.33)^k]$$
$$\sum_{i=0}^{k-1}(1/0.7)^i = \sum_{i=0}^{k-1}(1.43)^i = \frac{1-(1.43)^k}{1-1.43} = -2.33[1-(1.43)^k]$$

Putting the above expression in $y(k)$

$$y(k) = 0.475(-0.3)^k - 5.3(-0.7)^k + (-0.3)^k(3.33)^k + 5.825(-0.7)^{k-1}(1.43)^k$$

Controllability and observability are two important properties of state models which are to be studied prior to designing a controller.

Controllability deals with the possibility of forcing the system to a particular state by application of a control input. If a state is uncontrollable then no input will be able to control that state. On the other hand whether or not the initial states can be observed from the output is determined using observability property. Thus if a state is not observable then the controller will not be able to determine its behavior from the system output and hence not be able to use that state to stabilize the system.

1. Controllability

Before going to any details, we would first formally define controllability. Consider a dynamical system

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k) \end{aligned} \qquad (1)$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.

Definition 1 *The state equation (1) (or the pair (A,B)) is said to be completey state controllable or simply controllable if for any initial state x*(0) *and any final state x(N), there exists an input sequence* , $k = 0, 1, 2, \cdots, N$, *which transfers x(0) to x(N) for some finite N. Otherwise the state equation (1) is uncontrollable.*

Definition 2 ***Complete Output Controllability:*** *The system given in equation (1) is said to be completely output controllable or simply output controllable if any final output* $\boldsymbol{y}(N)$ *can be reached from any initial state* $\boldsymbol{x}(0)$ *by applying an unconstrained input* *sequence* $\boldsymbol{u}(k)$, $k = 0, 1, 2, \cdots, N$, *for some finite* N. *Otherwise (1) is not output controllable.*

1.1 Theorems on controllability

1. The state equation (1) or the pair (A,B) is controllable if and only if the $n \times nm$ controllability matrix

$$U_C = \begin{bmatrix} B & AB & A^2B & \ldots\ldots & A^{n-1}B \end{bmatrix}$$

 has rank *n*, i.e., full row rank.

 2. The state equation (1) is controllable if the $n \times n$ controllability grammian matrix

$$W_c = \sum_{i=0}^{N-1} A^i B B^T (A^i)^T = \sum_{i=0}^{N-1} A^{N-1-i} B B^T (A^{N-1-i})^T$$

is nonsingular for any nonzero finite *N*.

 3. If the system has a single input and the state model is in controllable canonical form then the system is controllable.

 4. When *A* has distinct eigenvalues and in Jordan/Diagonal canonical form the state model is controllable if and only if all the rows of *B* are nonzero.

 5. When *A* has multiple order eigenvalues and in Jordan canonical form, then the state model is controllable if and only if

 i. each Jordan block corresponds to one distinct eigenvalue and

 ii. the elements of *B* that correspond to last row of each Jordan block are not all zero.

Output Controllability: The system in equation (1) is completely output controllable if and only if the $p \times (n+1)m$ output controllability matrix

$$U_{OC} = \begin{bmatrix} D & CB & CAB & CA^2B & \ldots\ldots & CA^{n-1}B \end{bmatrix}$$

has rank p, i.e., full row rank.

➢ Controllability to the origin and Reachability

There exist three different definitions of controllability in the literature:

1. Input transfers any state to any state. This definition is adopted in this course.

2. Input transfers any state to zero state. This is called controllability to the origin.

3. Input transfers zero state to any state. This is referred as controllability from the origin or reachability.

Above three definitions are equivalent for continuous time system. For discrete time systems definitions (1) and (3) are equivalent but not the second one.

Example: Consider the system $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$, $y(k) = Cx(k)$. where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } C = [0 \;\; 1]$$

Show if the system is controllable. Find the transfer function $\frac{Y(z)}{U(z)}$. Can you see any connection between controllability and the transfer function?
Solution: The controllability matrix is given by

$$U = [B \;\; AB] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Its determinant $\|U\| = 0 \Rightarrow U$ has a rank 1 which is less than the order of the matrix, i.e., 2. Thus the system is not controllable. The transfer function

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B \qquad = [0\;1]\begin{bmatrix} z+2 & -1 \\ -1 & z+2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{z+1}$$

Although state model is of order 2, the transfer function has order 1. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -3$. This implies that the transfer function is associated with pole-zero cancellation for the pole at -3. Since one of the dynamic modes is cancelled, the system became uncontrollable.

Observability

Definition 2 *The state model (1) (or the pair (A,C)) is said to be observable if any initial state x(0)can be uniquely determined from the knowldge of output y(k) and input sequence u(k), for* $k = 0, 1, 2, \cdots, N$*, where N is some finite time. Otherwise the state model (1) is unobservable.*

Theorems on observability

1. The state model (1) or the pair *(A,C)* is observable if and only if the $np \times n$ observability matrix

$$U_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank *n*, i.e., full column rank.

2. The state model (1) is observable if the $n \times n$ observability grammian matrix

$$W_O = \sum_{i=0}^{N-1} (A^i)^T C^T C A^i = \sum_{i=0}^{N-1} (A^{N-1-i})^T C^T C A^{N-1-i}$$

is nonsingular for any nonzero finite *N*.

3. If the state model is in observable canonical form then the system is observable.

4. When *A* has distinct eigenvalues and in Jordan/Diagonal canonical form, the state model is observable if and only if none of the columns of *C* contain zeros.

5. When *A* has multiple order eigenvalues and in Jordan canonical form, then the state model is observable if and only if

i. each Jordan block corresponds to one distinct eigenvalue and

ii. the elements of *C* that correspond to first column of each Jordan block are not all zero.

- **Theorem of Duality**

The pair (A,B) is controllable if and only if the pair (A^T,B^T) is observable.

Exercise: Prove the theorem of duality.

- **Loss of controllability or observability due to pole-zero cancellation**

We have already seen through an example that a system becomes uncontrollable when one of the modes is cancelled. Let us take another example.

Example:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [1 \quad 1]\boldsymbol{x}(k) \end{aligned}$$

The controllability matrix

$$U_C = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

implies that the state model is controllable. On the other hand, the observability matrix

$$U_O = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

has a rank 1 which implies that the state model is unobservable. Now, if we take a different set of state variables so that, $\bar{x}_1(k) = y(k)$, then the state variable model will be:

$$\begin{aligned} \bar{x}_1(k+1) &= y(k+1) \\ \bar{x}_1(k+2) &= y(k+2) = -y(k) - 2y(k+1) + u(k+1) + u(k) \end{aligned}$$

Lets us take $\bar{x}_2(k) = y(k+1) - u(k)$. The new state variable model is:

$$\begin{aligned} \bar{x}_1(k+1) &= \bar{x}_2(k) + u(k) \\ \bar{x}_2(k+1) &= -\bar{x}_1(k) - 2\bar{x}_2(k) - u(k) \end{aligned}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = [1 \quad 0]$$

which implies

The controllability matrix

$$\bar{U}_C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

implies that the state model is uncontrollable. The observability matrix

$$\bar{U}_O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

implies that the state model is observable. The system difference equation will result in a tranfer function which would involve pole-zero cancellation. Whenever there is a pole zero cancellation, the state space model will be either uncontrollable or unobservable or both.

- **Controllability/Observability after sampling**

Question: If a continuous time system is undergone a sampling process will its controllability or observability property be maintained?

The answer to the question depends on the sampling period *T* and the location of the Eigen values of *A*.

. Loss of controllability and/or observability occurs only in presence of oscillatory modes of the system.

. A sufficient condition for the discrete model with sampling period *T* to be controllable is that whenever $Re[\lambda_i - \lambda_j] = 0$, $|Im[\lambda_i - \lambda_j]| \neq 2\pi m/T$ for $m = 1, 2, 3, \ldots$

. The above is also a necessary condition for a single input case.

Note: *If a continuous time system is not controllable or observable, then its discrete time version, with any sampling period, is not controllable or observable.*

Unit – IV

Stability Analysis

Mapping between the S–Plane and the Z–Plane – Primary strips and Complementary Strips – Stability criterion – Modified routh's stability criterion and jury's stability test

Unit Objectives:

After reading this Unit, you should be able to understand:

- To examine the stability of the system using different tests. To study the conventional method of analyzing digital control systems in the w–plane

Unit Outcomes:

- The learner understands the stability of digital control systems and how to make the unstable system to stable

- **Relationship between s-plane and z-plane**

In the analysis and design of continuous time control systems, the pole-zero configuration of the transfer function in s-plane is often referred. We know that:

. Left half of s-plane $\Rightarrow$ Stable region.

. Right half of s-plane $\Rightarrow$ Unstable region.

For relative stability again the left half is divided into regions where the closed loop transfer function poles should preferably be located.

Similarly, the poles and zeros of a transfer function in z-domain govern the performance characteristics of a digital system.

One of the properties of *F*(s)* is that it has an infinite number of poles, located periodically with intervals of $\pm m w_s$ with *m* = 0, 1, 2,......, in the s-plane where w_s is the sampling frequency in rad/sec.

If the primary strip is considered, the path, as shown in Figure 1, will be mapped into a unit circle in the z-plane, centered at the origin.

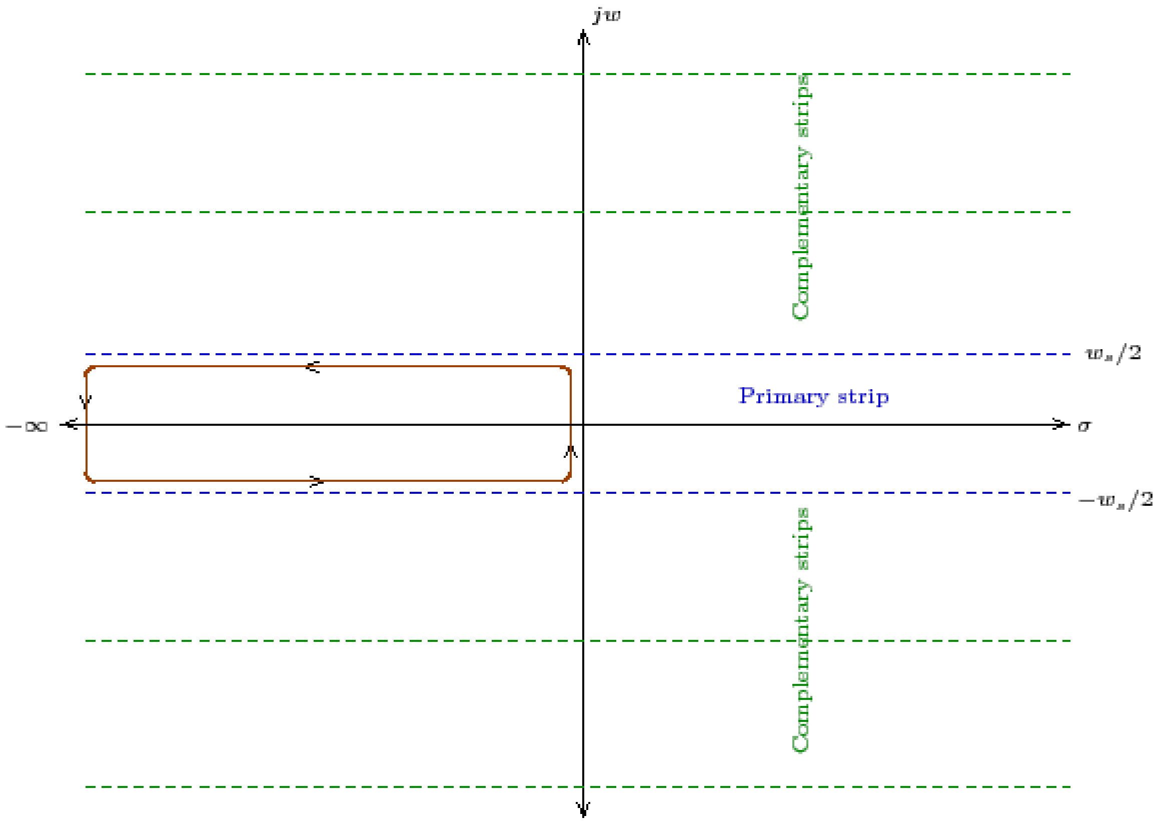


Figure 1: Primary and complementary strips in s-plane

The mapping is shown in Figure 2.

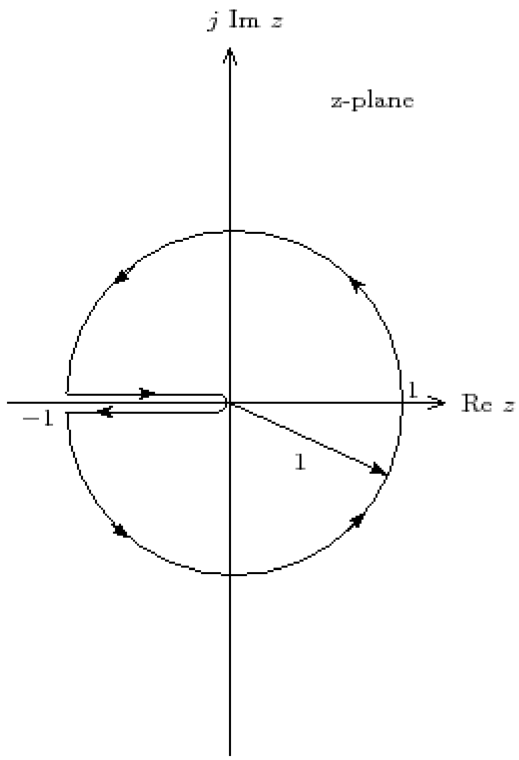


Figure 2: Mapping of primary strip in z-plane

Since

$$e^{(s+jmw_s)T} = e^{Ts}e^{j2\pi m}$$

$$= e^{Ts}$$

$$= z$$

where m is an integer, all the complementary strips will also map into the unit circle.

- **Mapping guidelines**

1. All the points in the left half s-plane correspond to points inside the unit circle in z-plane.
2. All the points in the right half of the s-plane correspond to points outside the unit circle.
3. Points on the *jw* axis in the s-plane correspond to points on the unit circle $|z| = 1$ in the z-plane.

$$s = jw$$

$$z = e^{Ts}$$

$$= e^{jwT} \Rightarrow$$

magnitude =1

- **Constant damping loci, constant frequency loci and contant damping ration loci**
- **Constant damping loci:** The real part **σ** of a pole, $s = \sigma + jw$, of a transfer function in s-domain, determines the damping factor which represents the rate of rise or decay of time response of the system.

. Large **σ** represents small time constant and thus a faster decay or rise and vice versa.

. The loci in the left half s-plane (vertical line parallel to *jw* axis as in Figure 2(a)) denote positive damping since the system is stable

. The loci in the right half s-plane denote negative damping.

. Constant damping loci in the z-plane are concentric circles with the center at $z = 0$, as shown in Figure 2(b).

. Negative damping loci map to circles with radii >1 and positive damping loci map to circles with radii <1.

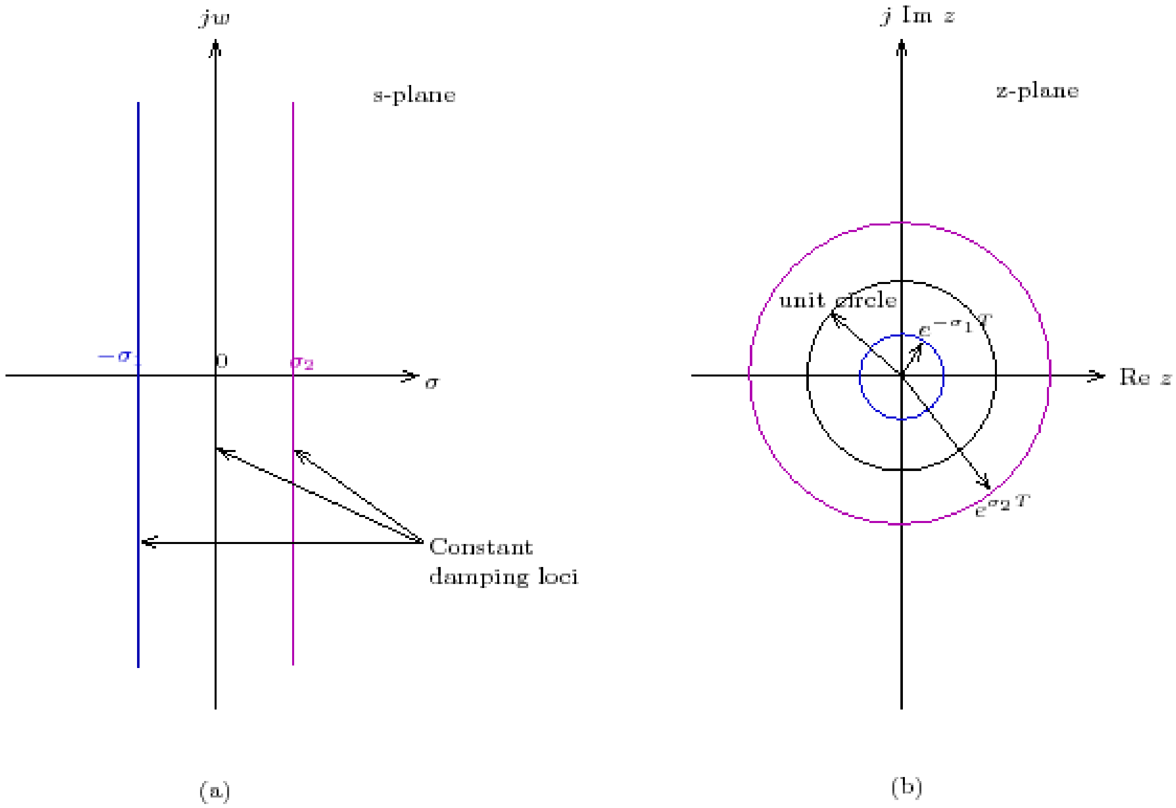


Figure 2: Constant damping loci in (a) s-plane and (b) z-plane

- **Constant frequency loci:** These are horizontal lines in s-plane, parallel to the real axis as shown in Figure 3(a).

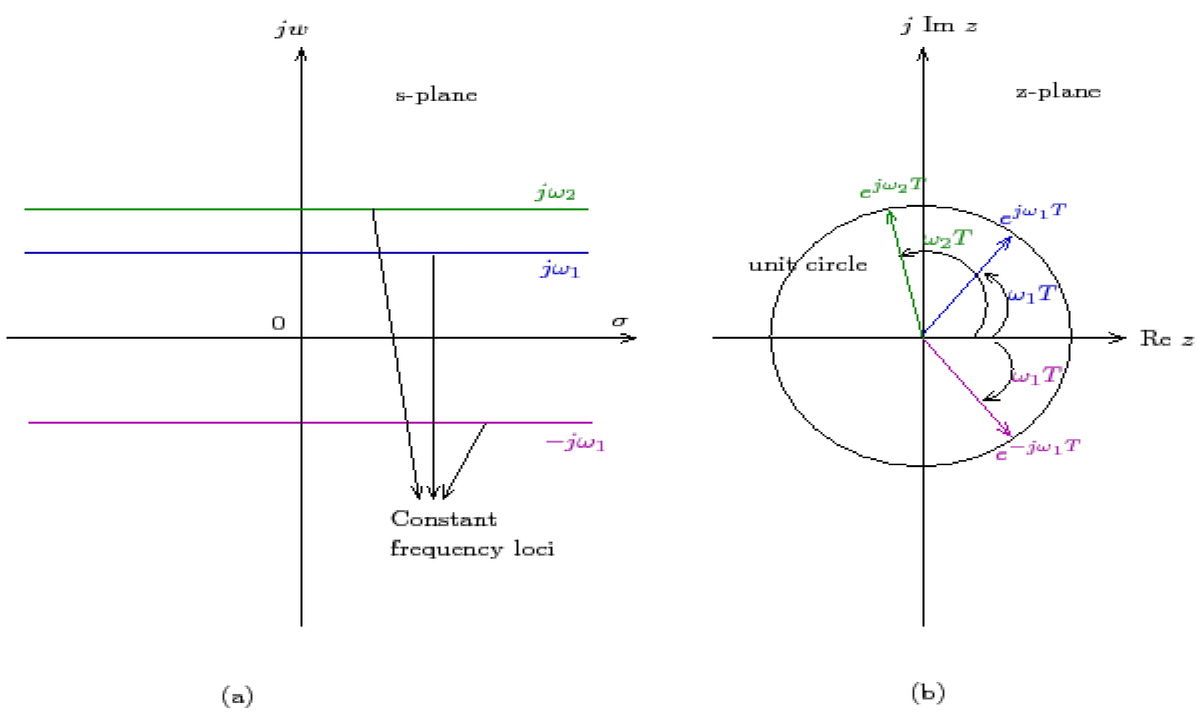


Figure 3: Constant frequency loci in (a) s-plane and (b) z-plane

Corresponding Z-transform:

$$z = e^{Ts}$$
$$= e^{jwT}$$

When w = constant, it represents a straight line from the origin at an angle of $\theta = wT$ rad, measured from positive real axis as shown in Figure 3 (b).

- **Constant damping ratio loci:** If ξ denotes the damping ratio:

$$s = -\xi w_n \pm jw_n\sqrt{1-\xi^2}$$
$$= -\frac{w}{\sqrt{1-\xi^2}}\xi \pm jw$$
$$= -w\tan\beta \pm jw$$

w_n is the natural undamped frequency and $\beta = \sin^{-1}\xi$. If we take Z-transform

$$z = e^{T(-w tan\beta + jw)}$$
$$= e^{-2\pi w \tan\beta / w_s} \angle(2\pi w / w_s)$$

For a given ξ or β, the locus in s-plane is shown in Figure 4(a).

In z-plane, the corresponding locus will be a logarithmic spiral as shown in Figure 4(b), except for $\xi = 0$ or $\beta = 0^o$ and $\xi = 1$ or $\beta = 90^o$.

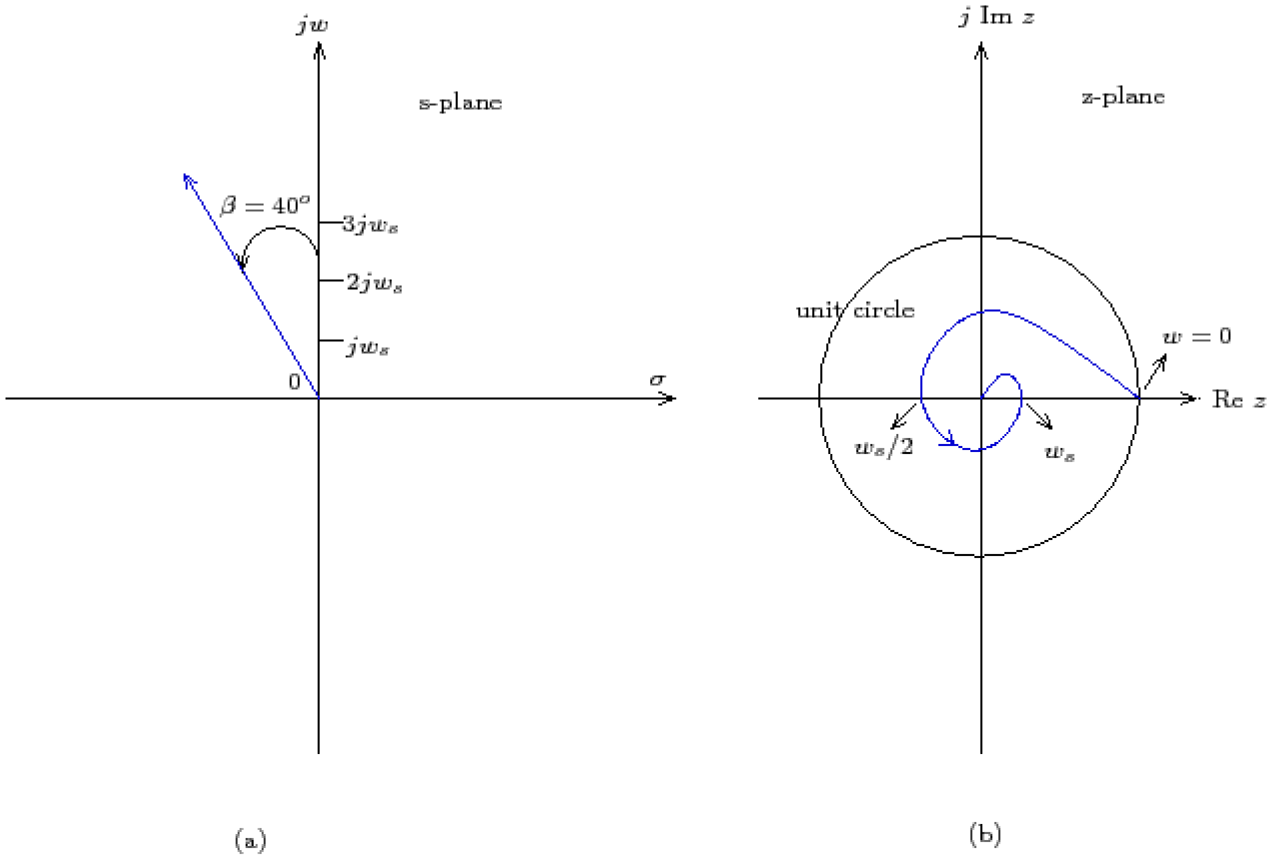


Figure 4: Constant damping ratio locus in (a) s-plane and (b) z-plane

Stability Analysis of closed loop system in z-plane

Stability is the most important issue in control system design. Before discussing the stability test let us first introduce the following notions of stability for a linear time invariant (LTI) system.

1. BIBO stability or zero state stbaility

2. Internal stability or zero input stability

Since we have not introduced the concept of state variables yet, as of now, we will limit our discussion to BIBO stability only.

An initially relaxed (all the initial conditions of the system are zero) LTI system is said to be BIBO stable if for every bounded input, the output is also bounded.

However, similar to continuous time systems, the stability of the following closed loop system

$$\frac{C(z)}{R(Z)} = \frac{G(z)}{1 + GH(z)}$$

can also be determined from the location of closed loop poles in z-plane which are the roots of the characteristic equation

$$1 + GH(z) = 0$$

1. For the system to be stable, the closed loop poles or the roots of the characteristic equation must lie within the unit circle in z-plane. Otherwise the system would be unstable.

2. If a simple pole lies at $|z| = 1$, the system becomes marginally stable. Similarly if a pair of complex conjugate poles lie on the $|z| = 1$ circle, the system is marginally stable. Multiple poles at the same location on unit circle make the system unstable.

Example 1:

Determine the closed loop stability of the system shown in Figure 1 when *K* = 1.

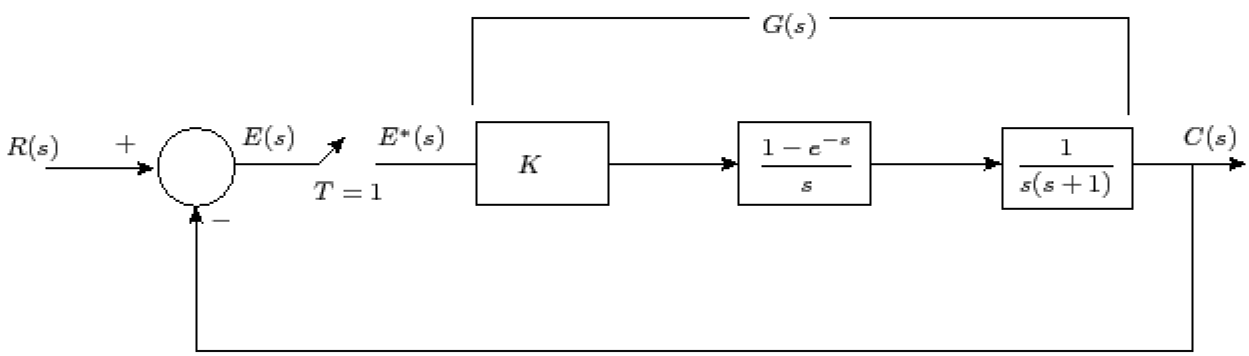


Figure 1: Example 1

Solution:

$$G(z) = Z\left[\frac{1 - e^{-s}}{s} . \frac{1}{s(s+1)}\right]$$

$$= (1 - z^{-1})Z\left[\frac{1}{s^2(s+1)}\right]$$

Since $H(s) = 1$, $G(z) = GH(z)$ and $\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$

$G(z)$ can be simplified as

$$G(z) = \frac{z-1}{z}.\left[\frac{z}{(z-1)^2} - \frac{(1-e^{-1})z}{(z-1)(z-e^{-1})}\right]$$

$$= \frac{(z-e^{-1})-(1-e^{-1})(z-1)}{(z-1)(z-e^{-1})}$$

$$= \frac{z-0.368-0.632z+0.632}{(z-1)(z-0.368)}$$

$$= \frac{0.368z+0.264}{(z-1)(z-0.368)}$$

We know that the characteristics equation is $\Rightarrow 1+G(z)=0$

$$\Rightarrow (z-1)(z-0.368)+0.368z+0.264=0$$

$$\Rightarrow z^2-z+0.632=0$$

$$\Rightarrow z_1=0.5+0.618j$$

$$\Rightarrow z_2=0.5-0.618j$$

Since $|z_1|=|z_2|$ <1, the system is stable.

Three stability tests can be applied directly to the characteristic equation without solving for the roots.

→ Schur-Cohn stability test
→ Jury Stability test
→ Routh stability coupled with bi-linear transformation.

Other stability tests like Lyapunov stability analysis are applicable for state space system models which will be discussed later. Computation requirements in Jury test is simpler than Schur-Cohn when the co-efficients are real which is always true for physical systems.

Jury Stability Test

Assume that the characteristic equation is as follows,

$P(z)=a_0z^n+a_1z^{n-1}+...+a_{n-1}z+a_n$, where $a_0>0$.

Jury Table

Row	z^0	z^1	z^2	z^3	z^4	...	z^n
1	a_n	a_{n-1}	$a_{n-2}\dots$	...	...		a_0
2	a_0	a_1	$a_2\dots$	...	...	...	a_n
3	b_{n-1}	b_{n-2}	...	...	b_0		
4	b_0	b_1	...	...	...	b_{n-1}	
5	c_{n-2}	$c_{n-3}\dots$	c_0				
6	c_0	$c_1\dots$	...	c_{n-2}			
.	....						
.	....						
.	....						
$2n-3$	q_2	q_1	q_0				

where,

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix}$$

$$k = 0, 1, 2, 3, ..., n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix}$$

$$k = 0, 1, 2, 3, ..., n-2$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}$$

This system will be stable if:

1. $|a_n| < a_0$
2. $P(z)|_{z=1} > 0$
3. $P(z)|_{z=-1} > 0$ for n even and $P(z)|_{z=-1} < 0$ for n odd
4.

$$\begin{array}{ll} |b_{n-1}| & > |b_0| \\ |c_{n-2}| & > |c_0| \\ . & \ldots\ldots\ldots\ldots\ldots \\ . & \ldots\ldots\ldots\ldots \\ |q_2| & > |q_0| \end{array}$$

Example 2: The characteristic equation is

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

Thus, $a_0 = 1 \quad a_1 = -1.2 \quad a_2 = 0.07 \quad a_3 = 0.3 \quad a_4 = -0.08$

We will now check the stability conditions.

1. $|a_n| = |a_4| = 0.08 < a_0 = 1 \Rightarrow$ First condition is satisfied.

2. $P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0 \Rightarrow$ Second condition is satisfied.

3. $P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0 \Rightarrow$ Third condition is satisfied.

Next we will construct the Jury Table.

Jury Table

$$b_3 = \begin{vmatrix} a_n & a_0 \\ a_0 & a_n \end{vmatrix} = 0.0064 - 1 = -0.9936$$

$$b_2 = \begin{vmatrix} a_n & a_1 \\ a_0 & a_3 \end{vmatrix} = -0.08 \times 0.3 + 1.2 = 1.176$$

Rest of the elements are also calculated in a similar fashion. The elements are $b_1 = -0.0756 \quad b_0 = -0.204 \quad c_2 = 0.946 \quad c_1 = -1.184 \quad c_0 = 0.315$.

One can see

$$|b_3| = 0.9936 > |b_0| = 0.204$$

$$|c_2| = 0.946 > |c_0| = 0.315$$

All criteria are satisfied. Thus the system is stable.

Example 3: The characteristic equation is $P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$

Thus $a_0 = 1 \quad a_1 = -1.3 \quad a_2 = -0.08 \quad a_3 = 0.24$.

Stability conditions are:

1. $|a_3| = 0.24 < a_0 = 1 \Rightarrow$ First condition is satisfied.

2. $P(1) = 1 - 1.3 - 0.08 + 0.24 = -0.14 < 0 \Rightarrow$ Second condition is not satisfied.

Since one of the criteria is violated, we may stop the test here and conclude that the system is unstable. $P(1) = 0$ or $P(-1) = 0$ indicates the presence of a root on the unit circle and in that case the system can at the most become marginally stable if rest of the conditions are satisfied.

The stability range of a parameter can also be found from Jury's test which we will see in the next example.

Example 4: Consider the system shown in Figure 1. Find out the range of *K* for which the system is stable.

Solution:

$$G(z) = \frac{K(0.368z + 0.264)}{(z-1)(z-0.368)}$$

The closed loop transfer function:

$$\frac{C(z)}{R(z)} = \frac{K(0.368z + 0.264)}{z^2 + (0.368K - 1.368)z + 0.368 + 0.264K}$$

$$P(z) = z^2 + (0.368K - 1.368)z + 0.368 + 0.264K = 0$$

Characteristic equation:

Since it is a second order system only 3 stability conditions will be there.

1. $|a_2| < a_0$
2. $P(1) > 0$
3. $P(-1) > 0$ since $n = 2$=even. This implies:

1. $|0.368 + 0.264K| < 1 \quad \Rightarrow 2.39 > K > -5.18$
2. $P(1) = 1 + (0.368K - 1.368) + 0.368 + 0.264K = 0.632K > 0 \Rightarrow K > 0$
3. $P(-1) = 1 - (0.368K - 1.368) + 0.368 + 0.264K = 2.736 - 0.104K > 0 \Rightarrow 26.38 > K$

Combining all, the range of K is found to be $0 < K < 2.39$.

If $K = 2.39$, system becomes critically stable. The characteristics equation becomes:

$$z^2 - 0.49z + 1 = 0 \qquad \Rightarrow z = 0.244 \pm j0.97$$

Sampling period $T = 1$ sec.

$$w_d = \frac{w_s}{2\pi}\angle z = \frac{2\pi}{2\pi}\tan^{-1}\frac{0.97}{0.244} \cong 1.324$$

rad/sec

The above frequency is the frequency of sustained oscillation.

1.2 Singular Cases

The situation, when some or all of the elements of a row in the Jury table are zero, indicates the presence of roots on the unit circle. This is referred to as a singular case.

It can be avoided by expanding or contracting unit circle infinitesimally by an amount ε which is equivalent to move the roots of P(z) off the unit circle. The transformation is:

$$z_1 = (1 + \epsilon)z$$

where ε is a very small number. When ε is positive the unit circle is expanded and when ε is negative the unit circle is contracted. The difference between the number of zeros found inside or outside the unit circle when the unit circle is expanded or contracted is the number of zeros on the unit circle.

Since $(1+\epsilon)^n z^n \cong (1+n\epsilon)z^n$ for both positive and negative ε, the transformation requires the coefficient of the z^n term be multiplied by $(1+n\epsilon)$.

Example 5: The characteristic equation: $P(z) = z^3 + 0.25z^2 + z + 0.25 = 0$

Thus, $a_0 = 1 \quad a_1 = 0.25 \quad a_2 = 1 \quad a_3 = 0.25$.

We will now check the stability conditions.

1. $|a_n| = |a_3| = 0.25 < a_0 = 1 \Rightarrow$ First condition is satisfied.

2. $P(1) = 1 + 0.25 + 1 + 0.25 = 2.5 > 0 \Rightarrow$ Second condition is satisfied.

3. $P(-1) = -1 + 0.25 - 1 + 0.25 = -1.5 < 0 \Rightarrow$ Third condition is satisfied.

Jury Table

$$b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix} = \quad 0.0625 - 1 = -0.9375$$

$$b_1 = \begin{vmatrix} a_3 & a_1 \\ a_0 & a_2 \end{vmatrix} = \quad 0.25 - 0.25 = 0$$

Since the element b_1 is zero, we know that some of the roots lie on the unit circle.

If we replace z by (1 + ε) z, the characteristic equation would become:

$$(1+3\epsilon)z^3 + 0.25(1+2\epsilon)z^2 + (1+\epsilon)z + 0.25 = 0$$

First three stability conditions are satisfied when ε $\rightarrow 0^+$.

Jury Table

Row	z^0	z^1	z^2	z^3
1	0.25	$1+\epsilon$	$0.25(1+2\epsilon)$	$1+$
2	$1+3\epsilon$	$0.25(1+2\epsilon)$	$1+\epsilon$	0.2
3	$0.25^2-(1+3\epsilon)^2$	$0.25(1+\epsilon)-0.25(1+2\epsilon)(1+3\epsilon)$	$-(1+3\epsilon)(1+\epsilon)+0.25^2(1+2\epsilon)$	

$|b_2| = |0.0625-(1+6\epsilon+9\epsilon^2)|$ and $|b_0| = |0.0625-(1+3.875\epsilon+3\epsilon^2)|$.

Since, when $\varepsilon \to 0^+$, $1+6\epsilon+9\epsilon^2 > 1+3.875\epsilon+3\epsilon^2$, thus $|b_2| > |b_0|$ which implies that the roots which are not on the unit circle are actually inside it and the system is marginally stable. The roots of the characteristic equation are found out to be $\pm i$ and -0.25 which verifies our conclusion.

Stability Analysis using Bilinear Transformation and Routh Stability Criterion

Another frequently used method in stability analysis of discrete time system is the bilinear transformation coupled with Routh stability criterion. This requires transformation from z-plane to another plane called w-plane.

The bilinear transformation has the following form.

$$z = \frac{aw+b}{cw+d}$$

where a, b, c, d are real constants. If we consider $a = b = c = 1$ and $d = -1$, then the transformation takes a form

$$z = \frac{w+1}{w-1}$$

or,

$$w = \frac{z+1}{z-1}$$

This transformation maps the inside of the unit circle in the z-plane into the left half of the w-plane.

Let the real part of w be α and imaginary part be &beta.

$$\Rightarrow w = \alpha + j\beta$$

. The inside of the unit circle in z -plane can be represented by:

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\alpha + j\beta + 1}{\alpha + j\beta - 1}\right| < 1$$

$$\Rightarrow \frac{(\alpha+1)^2 + \beta^2}{(\alpha-1)^2 + \beta^2} < 1 \Rightarrow (\alpha+1)^2 + \beta^2 < (\alpha-1)^2 + \beta^2 \Rightarrow \alpha < 0$$

Thus inside of the unit circle in z -plane maps into the left half of w -plane and outside of the unit circle in z -plane maps into the right half of w -plane. Although w -plane seems to be similar to s -plane, quantitatively it is not same.

In the stability analysis using bilinear transformation, we first substitute $z = \frac{w+1}{w-1}$ in the characteristics equation $P(z) = 0$ and simplify it to get the characteristic equation in w -plane as $Q(w) = 0$. Once the characteristics equation is transformed as $Q(w) = 0$, Routh stability criterion is directly used in the same manner as in a continuous time system.

We will now solve the same examples which were used to understand the Jury's test.

Example 1 The characteristic equation:

$$P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$$

Transforming $P(z)$ into w -domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^3 - 1.3\left[\frac{w+1}{w-1}\right]^2 - 0.08\left[\frac{w+1}{w-1}\right] + 0.24 = 0$$

or,

$$Q(w) = 0.14w^3 - 1.06w^2 - 5.1w - 1.98 = 0$$

We can now construct the Routh array as

w^3	0.14	−5.1
w^2	−1.06	−1.98
w^1	−5.36	
w^0	−1.98	

There is one sign change in the first column of the Routh array. Thus the system is unstable with one pole at right hand side of the *w* -plane or outside the unit circle of *z* -plane.

Example2: The characteristic equation:

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

Transforming *P(z)* into *w* -domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^4 - 1.2\left[\frac{w+1}{w-1}\right]^3 + 0.07\left[\frac{w+1}{w-1}\right]^2 + 0.3\left[\frac{w+1}{w-1}\right] - 0.08 = 0$$

or,

$$Q(w) = 0.09w^4 + 1.32w^3 + 5.38w^2 + 7.32w + 1.89 = 0$$

We can now construct the Routh array as

w^4	0.09	5.38	1.89
w^3	1.32	7.32	
w^2	4.88	1.89	
w^1	6.81		
w^0	1.89		

All elements in the first column of Routh array are positive. Thus the system is stable.

Example 3:

Consider the system shown in Figure 1. Find out the range of *K* for which the system is stable.

Solution:

$$G(z) = \frac{K(0.084z^2 + 0.17z + 0.019)}{(z^3 - 1.5z^2 + 0.553z - 0.05)}$$

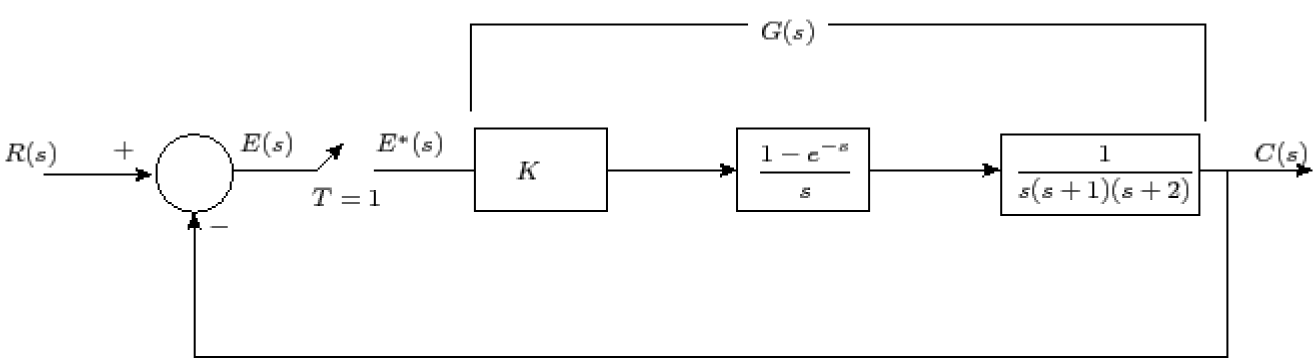


Figure 1: Figure for Example 3

Characteristic equation:

$$1 + \frac{K(0.084z^2 + 0.17z + 0.019)}{(z^3 - 1.5z^2 + 0.553z - 0.05)} = 0$$

$$P(z) = z^3 + (0.084K - 1.5)z^2 + (0.17K + 0.553)z + (0.019K - 0.05) = 0$$

Transforming *P(z)* into *w* -domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^3 + (0.084K - 1.5)\left[\frac{w+1}{w-1}\right]^2 + (0.17K + 0.553)\left[\frac{w+1}{w-1}\right] + (0.019K - 0.05) = 0$$

or,

$$Q(w) = (0.003 + 0.27K)w^3 + (1.1 - 0.11K)w^2 + (3.8 - 0.27K)w + (3.1 + 0.07K) = 0$$

We can now construct the Routh array as

$$\begin{array}{lll}
w^3 & 0.003 + 0.27K & 3.8 - 0.27K \\
w^2 & 1.1 - 0.11K & 3.1 + 0.07K \\
w^1 & \dfrac{0.01K^2 - 1.55K + 4.17}{1.1 - 0.11K} & \\
w^0 & 3.1 + 0.07K &
\end{array}$$

The system will be stable if all the elements in the first column have same sign. Thus the conditions for stability, in terms of *K*, are

$$0.003 + 0.27K > 0 \Rightarrow K > -0.011$$

$$1.1 - 0.11K > 0 \Rightarrow K < 10$$

$$0.01K^2 - 1.55K + 4.17 > 0 \Rightarrow K < 2.74 \quad \text{or,} \quad K > 140.98$$

$$3.1 + 0.07K > 0 \Rightarrow K > -44.3$$

Combining above four constraints, the stable range of *K* can be found as

$$-0.011 < K < 2.74$$

- **Singular Cases**

In Routh array, tabulation may end in occurance with any of the following conditions.

. The first element in any row is zero

. All the elements in a single row are zero.

The remedy of the first case is replacing zero by a small number ε and then proceeding with the tabulation. Stability can be checked for the limiting case. Second singular case indicates one or more of the following cases.

. Pairs of real roots with opposite signs.

. Pairs of imaginary roots.

. Pairs of complex conjugate roots which are equidistant from the origin.

When a row of all zeros occurs, an auxiliary equation $A(w)= 0$ is formed by using the coefficients of the row just above the row of all zeros. The roots of the auxiliary equation are also the roots of the characteristic equation. The tabulation is continued by replacing the row of zeros by the coefficients of $\frac{dA(w)}{dw}$.

Looking at the correspondence between *w* -plane and *z* -plane, when an all zero row occurs, we can conclude that following two scenarios are likely to happen.

. Pairs of real roots in the *z* -plane that are inverse of each other.

. Pairs of roots on the unit circle simultaneously.

Example 4:

Consider the characteristic equation

$$P(z) = z^3 - 1.7z^2 - z + 0.8 = 0$$

Transforming *P(z)* into *w* -domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^3 - 1.7\left[\frac{w+1}{w-1}\right]^2 - \left[\frac{w+1}{w-1}\right] + 0.8 = 0$$

or,

$$Q(w) = 0.9w^3 + 0.1w^2 - 8.1w - 0.9 = 0$$

The Routh array:

$$\begin{array}{lll} w^3 & 0.9 & -8.1 \\ w^2 & 0.1 & -0.9 \\ w^1 & 0 & 0 \end{array}$$

The tabulation ends here. The auxiliary equation is formed by using the coefficients of w^2 row, as:

$$A(w) = 0.1w^2 - 0.9 = 0$$

Taking the derivative,

$$\frac{dA(w)}{dw} = 0.2w$$

Thus the Routh tabulation is continued as

$$\begin{array}{lll} w^3 & 0.9 & -8.1 \\ w^2 & 0.1 & -0.9 \\ w^1 & 0.2 & 0 \\ w^0 & -0.9 & \end{array}$$

As there is one sign change in the first column, one of the roots is on the right hand side of the w -plane. This implies that one root in z -plane lies outside the unit circle.

To verify our conclusion, the roots of the polynomial $z^3 - 1.7z^2 - z + 0.8 = 0$, are foun out to be z = 0.5, z = -0.8 and z = 2. Thus one can see that z = 2 lies outside the unit circle and it is inverse of z = 0.5 which caused the all zero row in w -plane.

Unit – V

Design of Discrete-time Control Systems by Conventional Methods

UNIT SYLLABUS

Transient and steady state specifications – Design using frequency response in the w–plane for lag and led compensators – Root locus technique in the z– plane

Unit Objectives:

After reading this Unit, you should be able to understand:

- To study the conventional method of analyzing digital control systems in the w–plane. To study the design of state feedback control by "the pole placement method."

Unit Outcomes:

- The learner understands the stability of digital control systems and how to make the unstable system to stable. The leaner will understand about designing of systems by conventional methods like root locus and bode plot through bilinear transformation.

- **Time Response of discrete time systems**

Absolute stability is a basic requirement of all control systems. Apart from that, good relative stability and steady state accuracy are also required in any control system, whether continuous time or discrete time. **Transient response** corresponds to the system close loop poles and **steady state response** corresponds to the excitation poles or poles of the input function.

- **Time response specifications**

In many practical control systems, the desired performance characteristics are specified in terms of time domain quantities. Unit step input is the most commonly used in analysis purpose of a system since it is easy to generate and represents a sufficiently drastic change thus provides useful information on both transient and steady state response.

The transient response of a system depends on the initial conditions. It is a common practice to

consider the system initially at rest.
Consider the digital control system shown in Figure 1

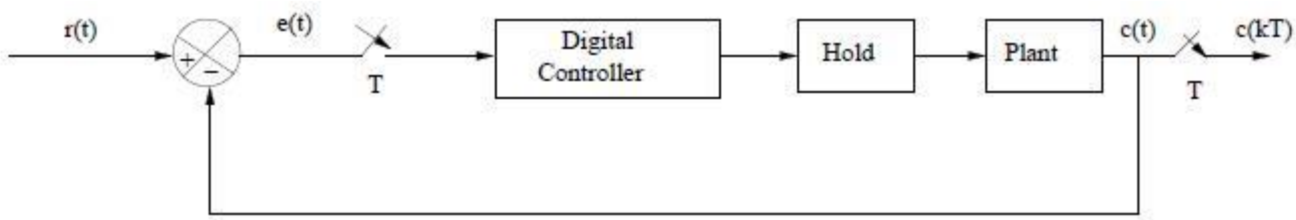


Figure 1: Block Diagram of a closed loop digital system

Similar to the continuous time case, transient response of a digital control system can also be characterized by the following.

1. Rise time (t_r): Time required for the unit step response to rise from 0% to 100% of its final value in case of underdamped system or 10% to 90% of its final value in case of overdamped system.

2. Delay time (t_d): Time required for the the unit step response to reach 50\% of its final value.

3. Peak time (t_p): Time at which maximum peak occurs.

4. Peak overshoot (M_p): The difference between the maximum peak and the steady state value of the unit step response.

5. Settling time (t_s): Time required for the unit step response to reach and stay within 2% or 5% of its steady state value. However since the output response is discrete the calculated performance measures may be slightly different from the actual values. Figure 2 illustrates this.

The output has a maximum value $c_{\max}$ whereas the maximum value of the discrete output is $c^*_{\max}$ which is always less than or equal to $c_{\max}$. If the sampling period is small enough compared to the oscillations of the response then this difference will be small otherwise $c^*_{\max}$ may be completely erroneous.

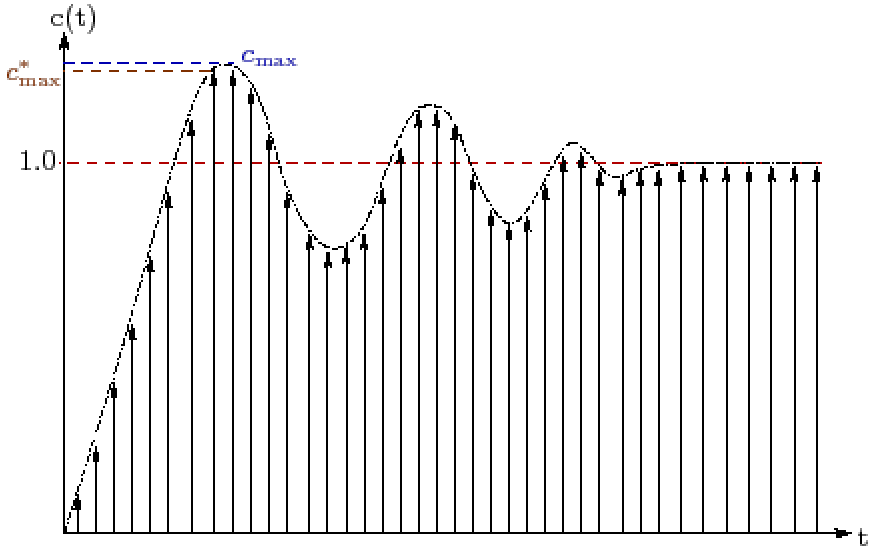


Figure 2: Unit step response of a discrete time system

- **Steady state error**

The steady state performance of a stable control system is measured by the steady error due to step, ramp or parabolic input depending on the system type. Consider the discrete time system as shown in Figure 3.

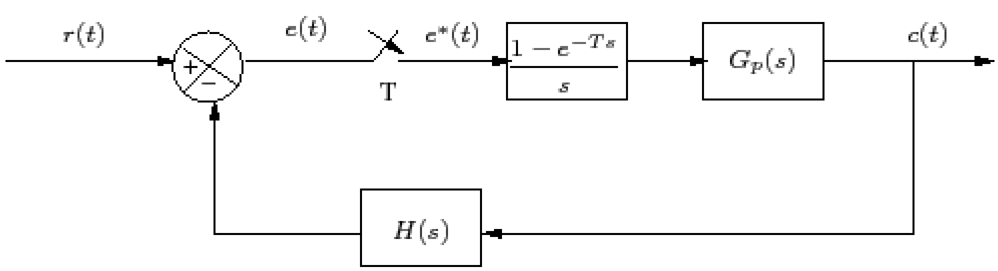


Figure 3: Block Diagram 2

From Figure 2, we can write

$$E(s) = R(s) - H(s)C(s)$$

We will consider the steady state error at the sampling instants. From final value theorem

$$
\begin{aligned}
\lim_{k\to\infty} e(kT) &= \lim_{z\to 1}\left[(1-z^{-1})E(z)\right]\\
G(z) &= (1-z^{-1})Z\left[\frac{G_p(s)}{s}\right]\\
GH(z) &= (1-z^{-1})Z\left[\frac{G_p(s)H(s)}{s}\right]\\
\frac{C(z)}{R(z)} &= \frac{G(z)}{1+GH(z)}\\
\text{Again, } E(z) &= R(z)-GH(z)E(z)\\
\text{or, } E(z) &= \frac{1}{1+GH(z)}R(z)\\
\Rightarrow e_{ss} &= \lim_{z\to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}R(z)\right]
\end{aligned}
$$

The steady state error of a system with feedback thus depends on the input signal $R(z)$ and the loop transfer function $GH(z)$.

- **Type-0 system and position error constant**

Systems having a finite nonzero steady state error with a zero order polynomial input (step input) are called **Type-0** systems. The position error constant for a system is defined for a step input.

$$
\begin{aligned}
r(t) &= u_s(t) \quad \text{unit step input}\\
R(z) &= \frac{1}{1-z^{-1}}\\
e_{ss} &= \lim_{z\to 1}\frac{1}{1+GH(z)} = \frac{1}{1+K_p}
\end{aligned}
$$

where $K_p = \lim_{z\to 1} GH(z)$ is known as the **position error constant**.

- **Type-1 system and velocity error constant**

Systems having a finite nonzero steady state error with a first order polynomial input (ramp input) are called **Type-1** systems. The velocity error constant for a system is defined for a ramp input.

$$\begin{aligned} r(t) &= u_r(t) \quad \text{unit ramp} \\ R(z) &= \frac{Tz}{(z-1)^2} \quad = \frac{TZ^{-1}}{(1-Z^{-1})^2} \\ e_{ss} &= \lim_{z\to 1} \frac{T}{(z-1)GH(z)} \quad = \frac{1}{K_v} \end{aligned}$$

$$K_v = \frac{1}{T}\lim_{z\to 1}\left[(z-1)GH(z)\right]$$

where is known as the **velocity error constant**.

- **Type-2 system and acceleration error constant**

Systems having a finite nonzero steady state error with a second order polynomial input (parabolic input) are called **Type-2** systems. The acceleration error constant for a system is defined for a parabolic input.

$$\begin{aligned} R(z) &= \frac{T^2 z(z+1)}{2(z-1)^3} \quad = \frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3} \\ e_{ss} &= \frac{T^2}{2}\lim_{z\to 1} \frac{(z+1)}{(z-1)^2\left[1+GH(z)\right]} \quad = \frac{1}{\lim_{z\to 1}\frac{(z-1)^2}{T^2}GH(z)} = \frac{1}{K_a} \end{aligned}$$

$$K_a = \lim_{z\to 1} \frac{(z-1)^2}{T^2} GH(z)$$

where is known as the **acceleration error constant**.

Table1 shows the steady state errors for different types of systems for different inputs.

Table 1: Steady state errors

System	Step input	Ramp input	Parabolic input
Type-0	$\frac{1}{1+kp}$	∞	∞
Type-1	0	$\frac{1}{K_v}$	∞
Type-2	0	0	$\frac{1}{K_a}$

Example 1: Calculate the steady state errors for unit step, unit ramp and unit parabolic inputs for the system shown in Figure 4.

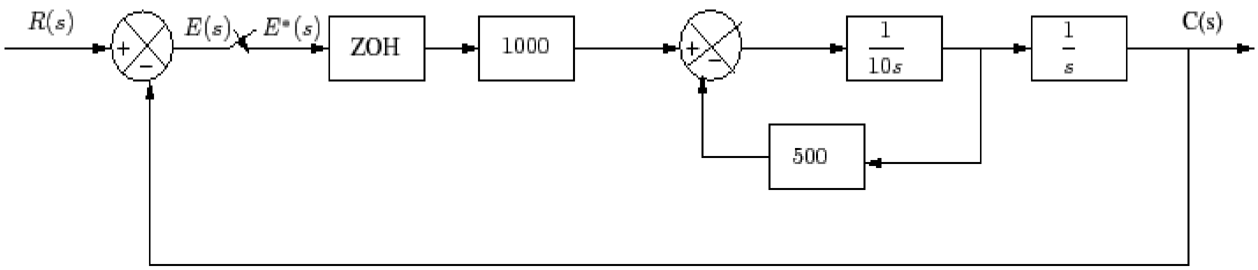


Figure 4: Block Diagram for Example 1

Solution: The open loop transfer function is:

$$\begin{aligned} G(s) &= \frac{C(s)}{E^*(s)} = G_{ho}(s)G_p(s) \\ &= \frac{1-e^{-Ts}}{s}\,\frac{1000/10}{s(s+500/10)} \end{aligned}$$

Taking Z-transform

$$\begin{aligned} G(z) &= 2(1-z^{-1})\ \mathrm{Z}\left[\frac{1}{s^2} - \frac{10}{500s} + \frac{10}{500(s+5000)}\right] \\ &= 2(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{10z}{500(z-1)} + \frac{10z}{500(z-e^{-50T})}\right] \\ &= \frac{1}{250}\left[\frac{(500T-10+10e^{-50T})z-(500T+10)e^{-50T}+10}{(z-1)(z-e^{-50T})}\right] \end{aligned}$$

Steady state error for step input $= \frac{1}{1+K_p}$ where $K_p = \lim_{z\to 1} G(z) = \infty \Rightarrow e_{ss}^{step} = 0$.

Steady state error for ramp input $= \frac{1}{K_v}$ where $K_v = \frac{1}{T}\lim_{z\to 1}[(z-1)G(z)] = 2$. $\Rightarrow e_{ss}^{ramp} = 0.5$.

Steady state error for parabolic input $= \frac{1}{K_a}$ where $K_a = \frac{1}{T^2}\lim_{z\to 1}[(z-1)^2G(z)] = 0$. $\Rightarrow e_{ss}^{para} = \infty$.

1 Prototype second order system

The study of a second order system is important because many higher order system can be approximated by a second order model if the higher order poles are located so that their contributions to transient response are negligible. A standard second order continuous time system is shown in Figure 1.

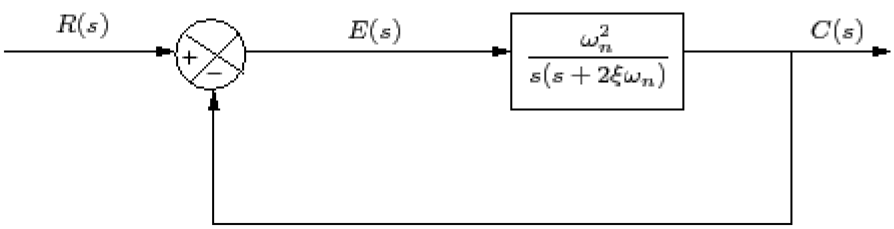


Figure 1: Block Diagram of a second order continuous time system

We can write,

$$\begin{aligned} G(s) &= \frac{\omega_n^2}{s(s+2\xi\omega_n)} \\ \text{Closed loop: } G_c(s) &= \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \\ \text{where, } \quad \xi &= \text{damping ratio} \\ \omega_n &= \text{natural undamped frequency} \\ \text{Roots: } & -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \end{aligned}$$

- **Bode Plot**

Bode plot is the graphical tool for drawing the frequency response of a system.

It is represented by two separate plots, one is the magnitude vs frequency and the other one is phase vs frequency. The magnitude is expressed in dB and the frequency is generally plotted in log scale.

One of the advantages of the Bode plot in s-domain is that the magnitude curve can be approximated by straight lines which allows the sketching of the magnitude plot without exact computation.

This feature is lost when we plot Bode diagram in z-domain. To incorporate this feature, we use bi-linear transformation to transform unit circle of the z-plane into the imaginary axis of another complex plane, *w* plane, where

$$w = \frac{1}{T}\ln(z)$$

From the power series expansion

$$w = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

$$\Rightarrow z = \frac{\frac{2}{T}+w}{\frac{2}{T}-w} = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$$

For frequency domain analysis the above bi-linear transformation may be used to convert *GH*(*z*) to*GH*(*w*) and then construct the Bode plot.

Example 1: Let us consider a digital control system for which the loop transfer function is given by

$$GH(z) = \frac{0.095z}{(z-1)(z-0.9)}$$

where sampling time *T* = 0.1 sec. Putting $z = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$, we get the transfer function in *w* plane as

$$GH(w) = \frac{10.02(1-0.0025w^2)}{w(1+1.0026w)} \qquad (T = 0.1\,\text{sec})$$

$$= \frac{10.02(1-0.05w)(1+0.05w)}{w(1+1.0026w)}$$

$$= \frac{10.02(1-0.05j\omega_w)(1+0.05j\omega_w)}{j\omega_w(1+j1.0026\omega_w)}$$

where ω_w is the frequency in *w* plane. Corner frequencies are 1/1.0026 = 0.997 rad/sec and 1/0.05 = 20 rad/sec.

The straight line asymptotes of the Bode plot can be drawn using the following.

- • Up to ω_w= 0.997 rad/sec, the magnitude plot is a straight line with slope - 20dB/decade. At ω_w= 0.01 rad/sec, the magnitude is $20\log_{10}(10.02) - 20\log_{10}(0.01) = 60$ dB.

• From ω_w= 0.997 rad/sec to ω_w= 20 rad/sec, the magnitude plot is a straight line with slope - 20 - 20 = - 40 dB/decade.

• Since both of the zeros will contribute same to the magnitude plot, after ω_w= 20 rad/sec, the slope of the straight line will be - 40 + 20 + 20 = 0 dB/decade.

The asymptotic magnitude plot is shown in Figure 1.

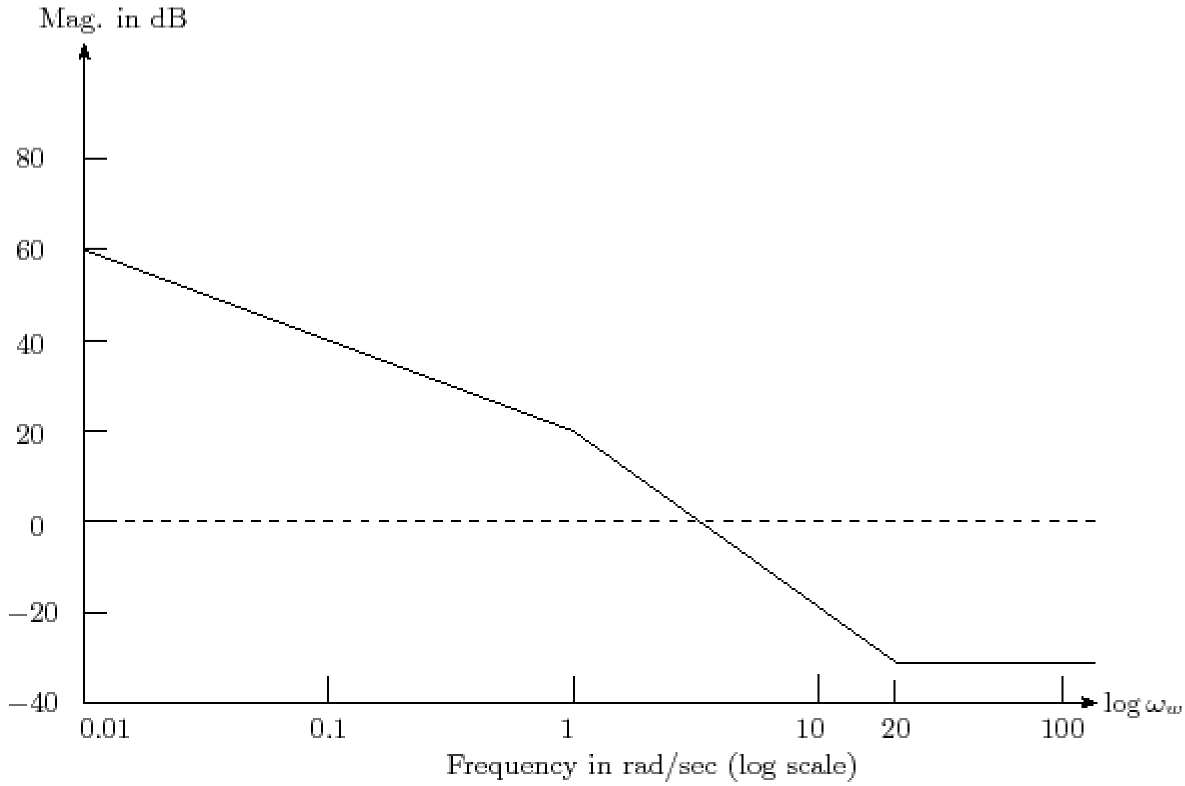


Figure 1: Bode asymptotic magnitude plot for Example 1

One should remember that the actual plot will be slightly different from the asymptotic plot. In the actual plot, errors due to straight line assumptions is compensated.

Phase plot is drawn by varying the frequency from 0.01 to 100 rad/sec at regular intervals. The phase angle contributed by one zero will be canceled by the other. Thus the phase will vary from

- 90° (270°) to - 180° (180°).

Figure 2 shows the actual magnitude and phase plot as drawn in MATLAB.

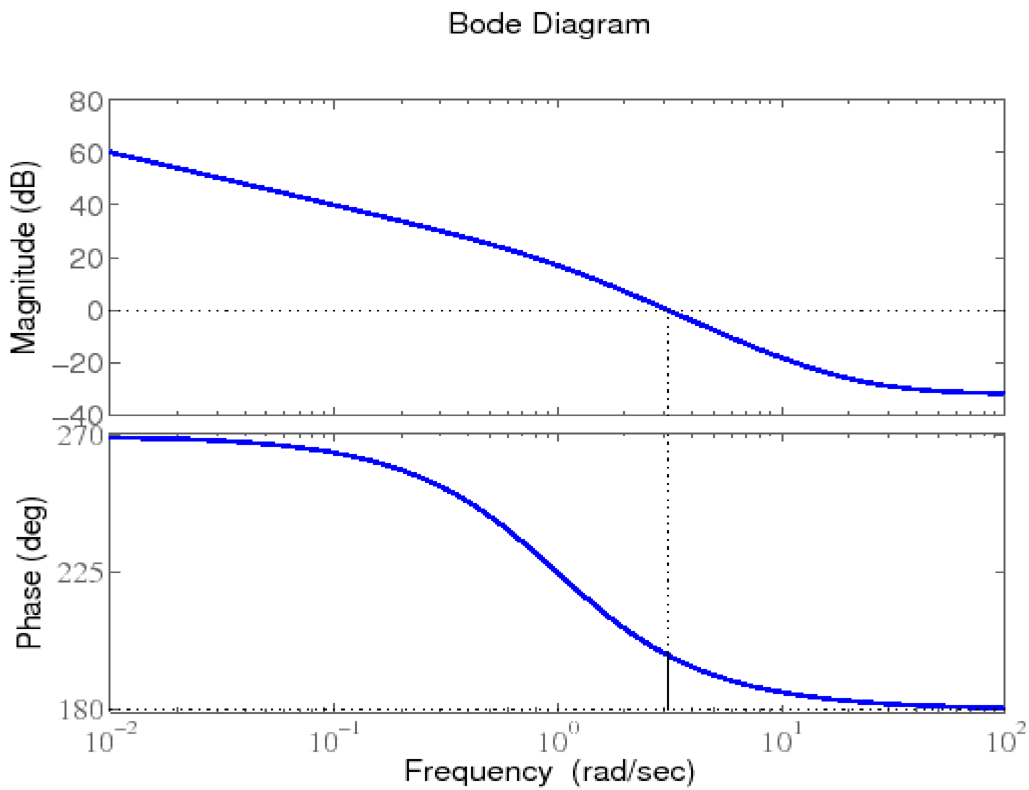


Figure 2: Bode magnitude and phase plot for Example 1

- **Gain margin and Phase margin**

Gain margin and phase margins are the measures of relative stability of a system.

Similar to continuous time case, we have to first define phase and gain cross over frequencies before defining gain margin and phase margin.

Gain margin is the safety factor by which the open loop gain of a system can be increased before the system becomes unstable. It is measured as

$$GM = 20\log_{10}\left|\frac{1}{GH(e^{i\omega_p T})}\right| dB$$

where ω_p is the phase crossover frequency which is defined as the frequency where the phase of the loop transfer function $GH(e^{i\omega T})$ is 180°.

Similarly, Phase margin (PM) is defined as

$$PM = 180^o + \angle GH(e^{i\omega_g T})$$

where ω_g is the gain crossover frequency which is defined as the frequency where the loop gain magnitude of the system becomes one.

- **Compensator design using Bode plot**

A compensator or controller is added to a system to improve its steady state as well as dynamic responses.

Nyquist plot is difficult to modify after introducing controller.

Instead Bode plot is used since two important design criteria, phase margin and gain crossover frequency are visible from the Bode plot along with gain margin.

Points to remember

- Low frequency asymptote of the magnitude curve is indicative of one of the error constants K_p, K_v, K_a depending on the system types.
- Specifications on the transient response can be translated into phase margin (PM), gain margin (GM), gain crossover frequency, bandwidth etc.
- Design using bode plot is simple and straight forward.
- Reconstruction of Bode plot is not a difficult task.

- **Phase lesd, Phase lag and Lag-lead compensators**

Phase lead, phase lag and lag-lead compensators are widely used in frequency domain design.

Before going into the details of the design procedure, we must remember the following.

- Phase lead compensation is used to improve stability margins. It increases system bandwidth thus improving the spread of the response.

- Phase lag compensation reduces the system gain at high frequencies with out reducing low frequency gain. Thus the total gain/low frequency gain can be increased which in turn will improve the steady state accuracy. High frequency noise can also be attenuated. But stability margin and bandwidth reduce.
- Using a lag lead compensator, where a lag compensator is cascaded with a lead compensator, both steady state and transient responses can be improved.
- Bi-linear transformation transfers the loop transfer function in z -plane to w -plane.

  Since qualitatively w -plane is similar to s -plane, design technique used in s -plane can be employed to design a controller in w -plane.

  Once the design is done in w -plane, controller in z -plane can be determined by using the inverse transformation from w -plane to z -plane.

  In the next two lectures we will discuss compensator design in s -plane and solve examples to design digital controllers using the same concept.

- **Compensator Design Using Bode Plot**

In this lecture we would revisit the continuous time design techniques using frequency domain since these can be directly applied to design for digital control system by transferring the loop transfer function in -plane to -plane.

- **Phase lead compensator**

If we look at the frequency response of a simple PD controller, it is evident that the magnitude of the compensator continuously grows with the increase in frequency.

The above feature is undesirable because it amplifies high frequency noise that is typically present in any real system.

In lead compensator, a first order pole is added to the denominator of the PD controller at frequencies well higher than the corner frequency of the PD controller.

A typical lead compensator has the following transfer function.

$$C(s) = K\frac{\tau s+1}{\alpha\tau s+1},$$ where, $$\alpha < 1$$

$\frac{1}{\alpha}$ is the ratio between the pole zero break point (corner) frequencies.

Magnitude of the lead compensator is $K\frac{\sqrt{1+\omega^2\tau^2}}{\sqrt{1+\alpha^2\omega^2\tau^2}}$. And the phase contributed by the lead compensator is given by

$$\phi = \tan^{-1}\omega\tau - \tan^{-1}\alpha\omega\tau$$

Thus a significant amount of phase is still provided with much less amplitude at high frequencies.

The frequency response of a typical lead compensator is shown in Figure 1 where the magnitude varies from $20\log_{10}K$ to $20\log_{10}\frac{K}{\alpha}$ and maximum phase is always less than 90° (around 60° in general).

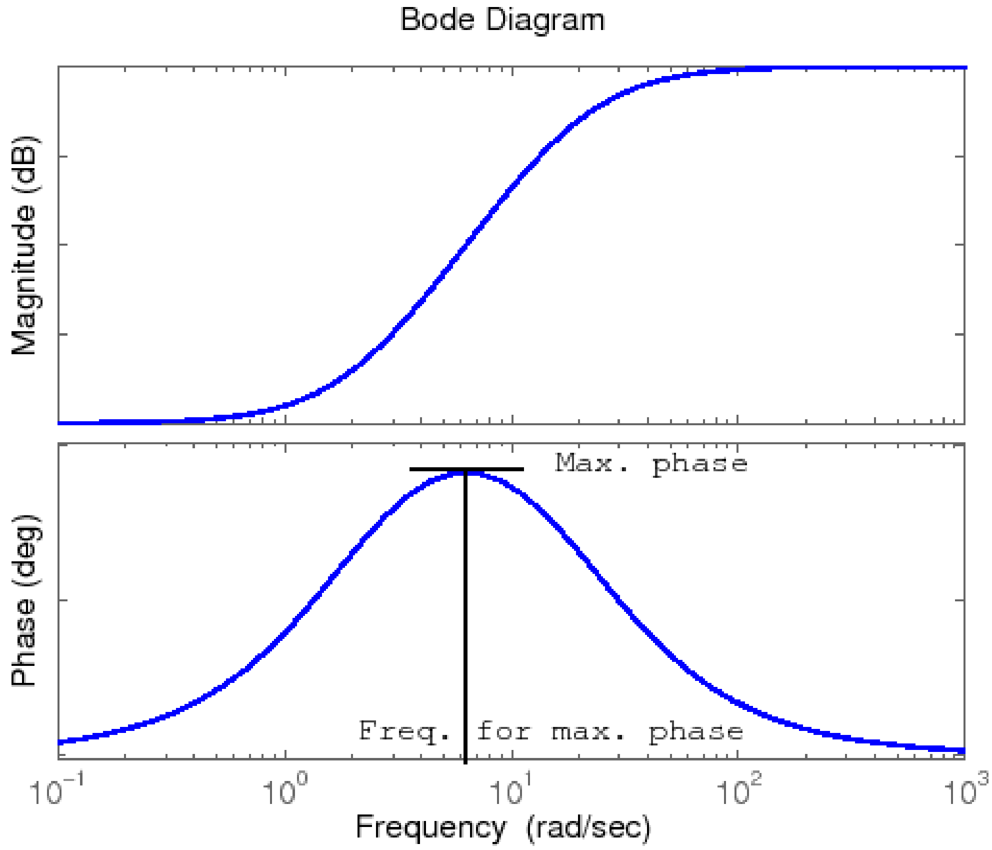


Figure 1: Frequency response of a lead compensator

It can be shown that the frequency where the phase is maximum is given by

$$\omega_{\max} = \frac{1}{\tau\sqrt{\alpha}}$$

The maximum phase corresponds to

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}$$

$$\Rightarrow \alpha = \left(\frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}\right)$$

The magnitude of $C(s)$ at $\omega_{\max}$ is $\dfrac{K}{\sqrt{\alpha}}$.

Example 1: Consider the following system

$$G(s) = \frac{1}{s(s+1)}, \quad H(s) = 1$$

Design a cascade lead compensator so that the phase margin (PM) is at least 45° and steady state error for a unit ramp input is ≤ 0.1 .

The lead compensator is

$$C(s) = K\frac{\tau s + 1}{\alpha \tau s + 1}, \qquad \text{where,} \quad \alpha < 1$$

When $s \to 0$, $C(s) \to K$.

Steady state error for unit ramp input is

$$\frac{1}{\lim_{s\to 0} sC(s)G(s)} = \frac{1}{C(0)} = \frac{1}{K}$$

Thus $\frac{1}{K} = 0.1$, or $K = 10$.

PM of the closed loop system should be 45°. Let the gain crossover frequency of the uncompensated system with K be ω_g .

$$\begin{aligned} G(j\omega) &= \frac{1}{j\omega(j\omega+1)} \\ Mag. &= \frac{1}{\omega\sqrt{1+\omega^2}} \\ Phase &= -90^o - \tan^{-1}\omega \\ \Rightarrow \frac{10}{\omega_g\sqrt{1+\omega_g^2}} &= 1 \\ \frac{100}{\omega_g^2(1+\omega_g^2)} &= 1 \\ \Rightarrow \omega_g &= 3.1 \end{aligned}$$

Phase angle at $\omega_g = 3.1$ is -90 - tan $^{-1}$ 3.1 = - 162° . Thus the PM of the uncompensated system with K is 18°.

If it was possible to add a phase without altering the magnitude, the additional phase lead required to maintain PM= 45° is 45° - 18° = 27° at ω_g = 3.1 rad/sec.

However, maintaining same low frequency gain and adding a compensator would increase the crossover frequency. As a result of this, the actual phase margin will deviate from the designed one. Thus it is safe to add a safety margin of ε to the required phase lead so that if it devaites also, still the phase requirement is met. In general ε is chosen between 5° to 15°.

So the additional phase requirement is 27° + 10° = 37° , The lead part of the compensator will provide this additional phase at ω_{max} .

Thus

$$\begin{aligned} \phi_{\max} &= 37^o \\ \Rightarrow \alpha &= \left(\frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}\right) = 0.25 \end{aligned}$$

The only parameter left to be designed is τ. To find τ, one should locate the frequency at which the uncompensated system has a logarithmic magnitude of $-20\log_{10}\frac{1}{\sqrt{\alpha}}$.

Select this frequency as the new gain crossover frequency since the compensator provides a gain of $20\log_{10}\frac{1}{\sqrt{\alpha}}$ at ω_{max}. Thus

$$\omega_{\max} = \omega_{g_{new}} = \frac{1}{\tau\sqrt{\alpha}}$$

In this case $\omega_{max} = \omega_{g\ new} = 4.41$. Thus

$$\tau = \frac{1}{4.41\sqrt{\alpha}} = 0.4535$$

The lead compensator is thus

$$C(s) = 10\,\frac{0.4535s+1}{0.1134s+1}$$

With this compensator actual phase margin of the system becomes 49.6° which meets the design criteria.

The corresponding Bode plot is shown in Figure 2

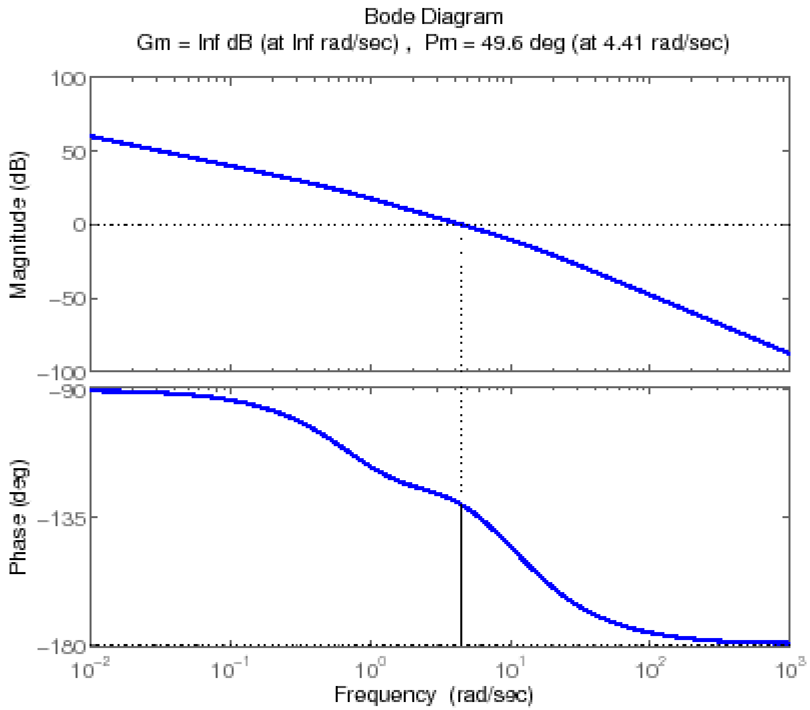


Figure 2: Bode plot of the compensated system for Example 1

Example2:

Now let us consider that the system as described in the previous example is subject to a sampled data control system with sampling time T = 0.2 sec. Thus

$$\begin{aligned} G_z(z) &= (1 - z^{-1})\mathrm{Z}\left[\frac{1}{s^2(s+1)}\right] \\ &= \frac{0.0187z + 0.0175}{z^2 - 1.8187z + 0.8187} \end{aligned}$$

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.1w)}{(1 - 0.1w)}$$

will transfer $G_z(z)$ into w -plane, as

$$G_w(w) = \frac{\left(1+\frac{w}{300}\right)\left(1-\frac{w}{10}\right)}{w(w+1)}$$

[please try the simplification]

We need first design a phase lead compensator so that PM of the compensated system is at least 50° with $K_v = 2$. The compensator in w -plane is

$$C(w) = K\frac{1+\tau w}{1+\alpha\tau w} \qquad 0<\alpha<1$$

Design steps are as follows.

K has to be found out from the K_v requirement.

Make $\omega_{max} = \omega_{gnew}$.

Compute the gain crossover frequency ω_g and phase margin of the uncompensated system after introducing K in the system. At ω_g check the additional/required phase lead, add safety margin, find out $\phi_{\max}$. Calculate α from the required $\phi_{\max}$ Since the lead part of the compensator provides a gain of $20\log_{10}\frac{1}{\sqrt{\alpha}}$, find out the frequency where the logarithmic magnitude is $-20\log_{10}\frac{1}{\sqrt{\alpha}}$. This will be the new gain crossover frequency where the maximum phase lead should occur. Calculate τ from the relation $\omega_{g_{new}} = \omega_{\max} = \frac{1}{\tau\sqrt{\alpha}}$

Now,

$$K_v = \lim_{w\to 0} wC(w)G_w(w) = 2$$

$$\Rightarrow K = 2$$

Using MATLAB command ``margin", phase margin of the system with $K = 2$ is computed as 31.6° with $\omega_g = 1.26$ rad/sec, as shown in Figure 3.

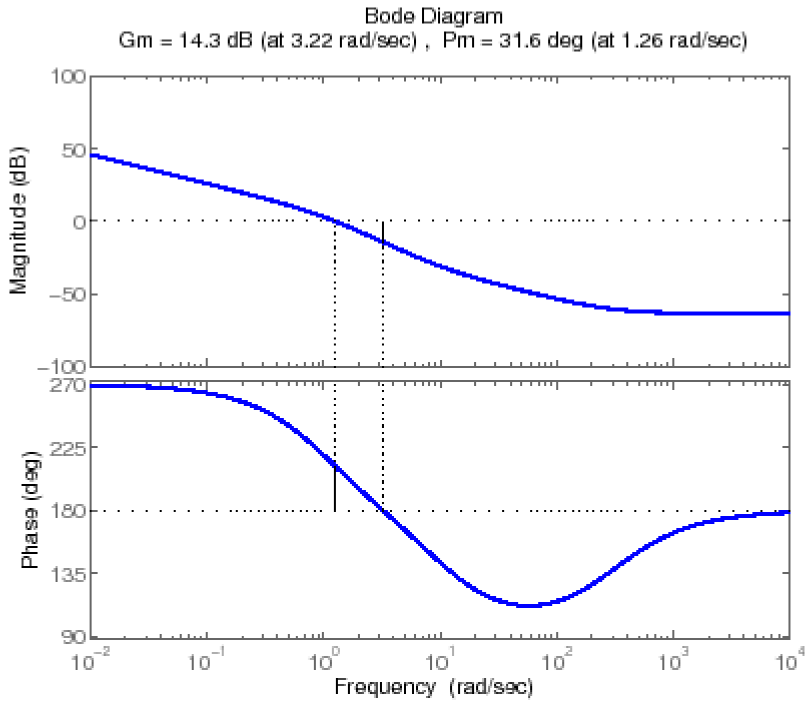


Figure 3: Bode plot of the uncompensated system for Example 2

Thus the required phase lead is 50° - 31.6° = 18.4° . After adding a safety margin of 11.6° , $\phi_{\max}$ becomes 30° . Hence

$$\alpha = \left(\frac{1-\sin(29^o)}{1+\sin(29^o)}\right) = 0.33$$

From the frequency response of the system it can be found that at $\omega = 1.75$ rad/sec, the magnitude of the system is $-20\log_{10}\frac{1}{\sqrt{\alpha}}$. Thus $\omega_{max} = \omega_{gnew} = 1.75$ rad/sec. This gives

$$1.75 = \frac{1}{\tau\sqrt{\alpha}}$$

Or,

$$\tau = \frac{1}{1.75\sqrt{0.33}} = 0.99$$

Thus the controller in *w*-plane is

$$C(w) = 2\frac{1+0.99w}{1+0.327w}$$

The Bode plot of the compensated system is shown in Figure 4.

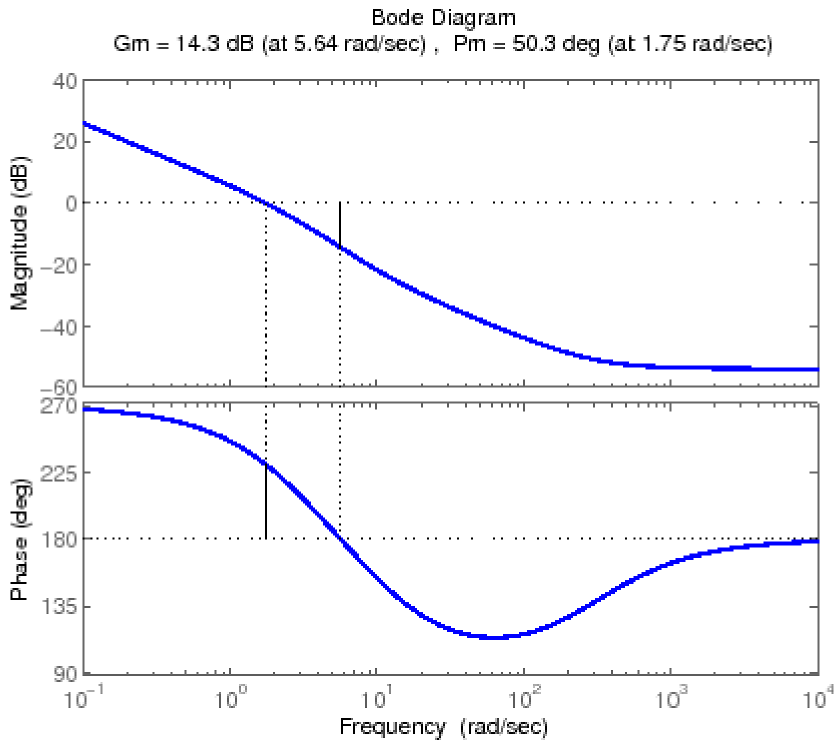


Figure 4: Bode plot of the compensated system for Example 2

Re-transforming the above controller into *z* -plane using the relation $w = 10\frac{z-1}{z+1}$, we get the controller in *z* -plane, as

$$C_z(z) \quad \cong \quad 2\,\frac{2.55z - 2.08}{z - 0.53}$$

- **Lag Compensator Design**

In the previous lecture we discussed lead compensator design. In this lecture we would see how to design a phase lag compensator

- **Phase lag compensator**

The essential feature of a lag compensator is to provide an increased low frequency gain, thus decreasing the steady state error, without changing the transient response significantly.

For frequency response design it is convenient to use the following transfer function of a lag compensator.

$$C_{lag}(s) = \alpha\frac{\tau s + 1}{\alpha\tau s + 1}, \quad \text{where,} \quad \alpha > 1$$

The above expression is only the lag part of the compensator. The overall compensator is

$$C(s) = KC_{lag}(s)$$

$$\text{when, } s \to 0, \;\; C_{lag}(s) \to \alpha$$
$$\text{when, } s \to \infty, \;\; C_{lag}(s) \to 1$$

Typical objective of lag compensator design is to provide an additional gain of α in the low frequency region and to leave the system with sufficient phase margin.

The frequency response of a lag compensator, with α=4 and τ=3, is shown in Figure 1 where the magnitude varies from $20\log_{10}\alpha$ dB to 0 dB.

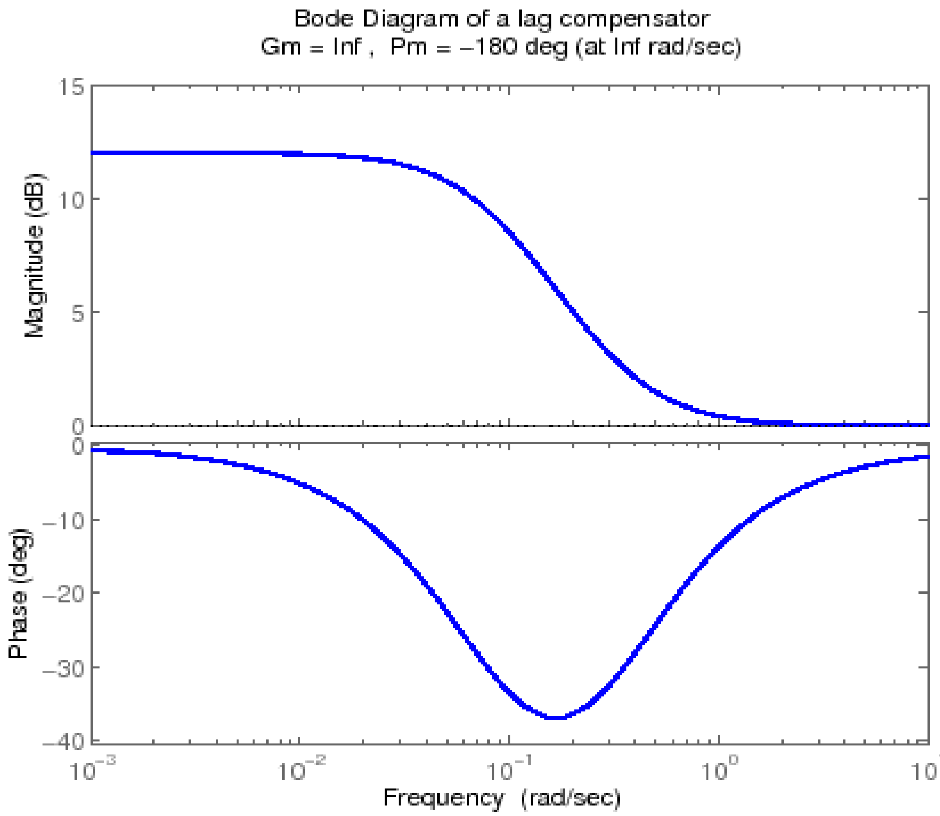


Figure 1: Frequency response of a lag compensator

Since the lag compensator provides the maximum lag near the two corner frequencies, to maintain the PM of the system, zero of the compensator should be chosen such that $\omega = 1/\tau$ is much lower than the gain crossover frequency of the uncompensated system.

In general, τ is designed such that $1/\tau$ is at least one decade below the gain crossover frequency of the uncompensated system. Following example will be comprehensive to understand the design procedure.

Example 1: Consider the following system

$$G(s) = \frac{1}{(s+1)(0.5s+1)}, \quad H(s) = 1$$

Design a lag compensator so that the phase margin (PM) is at least 50° and steady state error to a unit step input is ≤ 0.1.

The overall compensator is

$$C(s) = KC_{lag}(s) = K\alpha\frac{\tau s+1}{\alpha\tau s+1}, \text{ where, } \alpha > 1$$

When $s \to 0,\, C(s) \to K\alpha.$

Steady state error for unit step input is

$$\frac{1}{1+\lim_{s\to 0} C(s)G(s)} = \frac{1}{1+C(0)} = \frac{1}{1+K\alpha}$$

Thus, $\frac{1}{1+K\alpha} = 0.1, \text{ or, } K\alpha = 9.$

Now let us modify the system transfer function by introducing K with the original system. Thus the modified system becomes

$$G_m(s) = \frac{K}{(s+1)(0.5s+1)}$$

PM of the closed loop system should be 50°. Let the gain crossover frequency of the uncompensated system with K be ω_g.

$$\begin{aligned} G_m(j\omega) &= \frac{K}{(j\omega+1)(0.5j\omega+1)} \\ Mag. &= \frac{K}{\sqrt{1+\omega^2}\sqrt{1+0.25\omega^2}} \\ Phase &= -\tan^{-1}\omega - \tan^{-1}0.5\omega \end{aligned}$$

Required PM is 50°. Since the PM is achieved only by selecting K, it might be deviated from this value when the other parameters are also designed. Thus we put a safety margin of 5° to the PM which makes the required PM to be 55°.

$$\begin{aligned} \Rightarrow 180^o - \tan^{-1}\omega_g - \tan^{-1}0.5\omega_g &= 55^o \\ \text{or, } \tan^{-1}\frac{\omega_g + 0.5\omega_g}{1-0.5\omega_g^2} &= 125^o \\ \text{or, } \tan^{-1}\frac{1.5\omega_g}{1-0.5\omega_g^2} &= \tan 125^o = -1.43 \\ \text{or, } 0.715\omega_g^2 - 1.5\omega_g - 1.43 &= 0 \\ \Rightarrow \omega_g &= 2.8 \text{ rad/sec} \end{aligned}$$

To make ω_g = 2.8 rad/sec, the gain crossover frequency of the modified system, magnitude at ω_g should be 1. Thus

$$\frac{K}{\sqrt{1+\omega_g^2}\sqrt{1+0.25\omega_g^2}} = 1$$

Putting the value of ω_g in the last equation, we get K = 5.1. Thus,

$$\alpha = \frac{9}{K} = 1.76$$

The only parameter left to be designed is τ.

Since the desired PM is already achieved with gain K, We should place $\omega = 1/\tau$ such that it does not much effect the PM of the modified system with K. If we place $1/\tau$ one decade below the gain crossover frequency, then

$$\frac{1}{\tau} = \frac{2.8}{10}, \qquad \tau = 3.57$$

or,

The overall compensator is

$$C(s) = 9\,\frac{3.57s + 1}{6.3s + 1}$$

With this compensator actual phase margin of the system becomes 52.7°, as shown in Figure 2, which meets the design criteria.

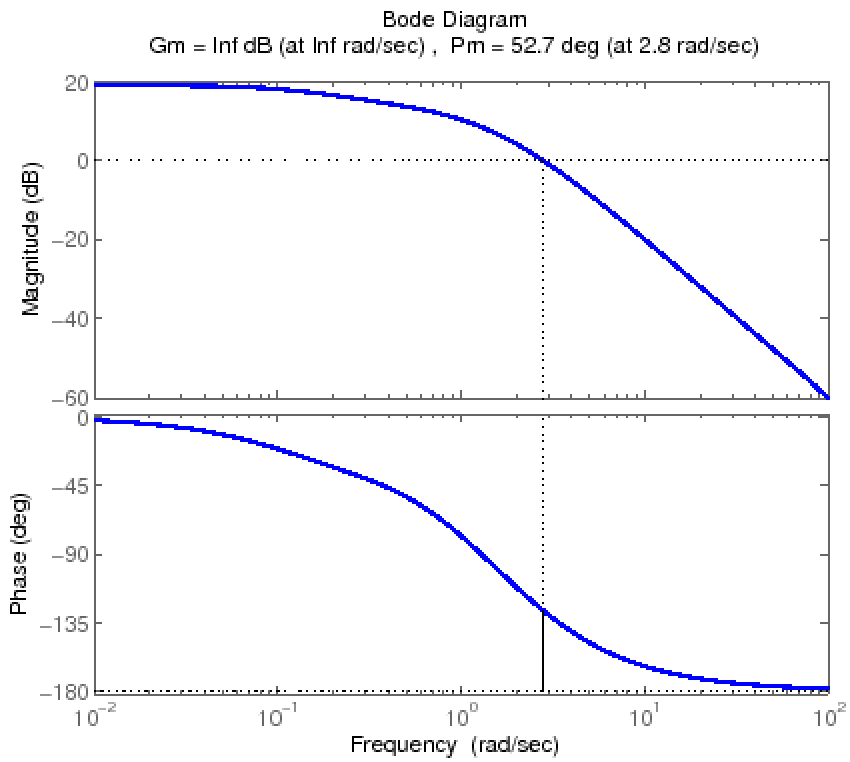


Figure 2: Bode plot of the compensated system for Example 1

Example2:

Now let us consider that the system as described in the previous example is subject to a sampled data control system with sampling time *T = 0.1* sec. We would use MATLAB to derive the plant transfer function *w* -plane.

Use the below commands.

>> s=tf('s');

>> gc=1/((s+1)*(0.5*s+1));

>> gz=c2d(gc,0.1,'zoh');

You would get

$$G_z(z) = \frac{0.009z + 0.0008}{z^2 - 1.724z + 0.741}$$

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.05w)}{(1 - 0.05w)}$$

will transfer $G_z(z)$ into *w*-plane. Use the below commands

>> aug=[0.1,1];

>> gwss = bilin(ss(gz),-1,'S_Tust',aug)

>> gw=tf(gwss)

to find out the transfer function in *w*-plane, as

$$\begin{aligned} G_w(w) &= \frac{1.992 - 0.09461w - 0.00023w^2}{w^2 + 2.993w + 1.992} \\ &\cong \frac{-0.00025(w-20)(w+400)}{(w+1)(w+2)} \end{aligned}$$

The Bode plot of the uncompensated system is shown in Figure 3.

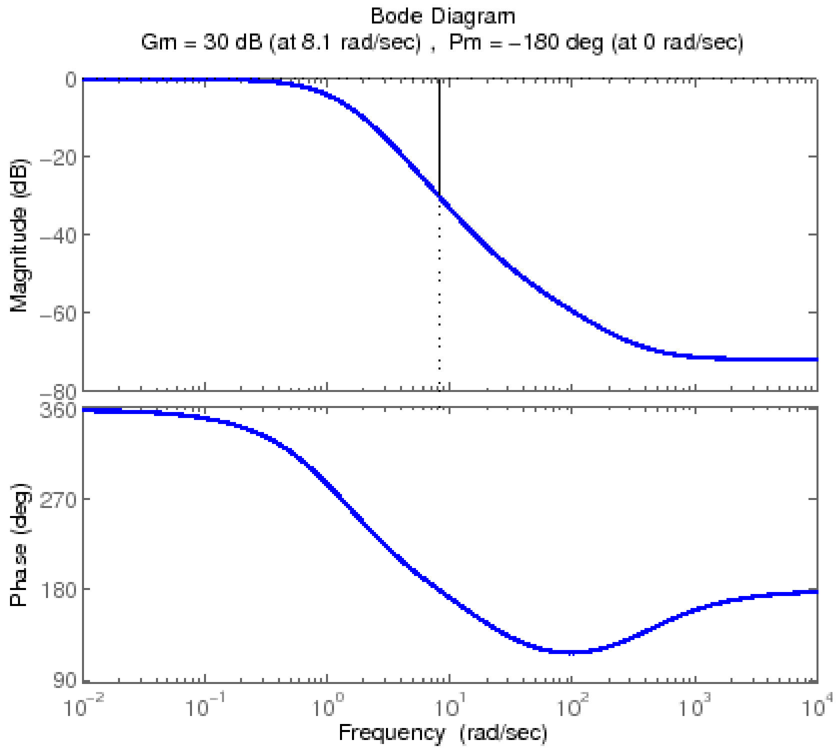


Figure 3: Bode plot of the uncompensated system for Example 2

We need to design a phase lag compensator so that PM of the compensated system is at least 50° and steady state error to a unit step input is 0.1. The compensator in *w* -plane is

$$C(w) = K\alpha\frac{1+\tau w}{1+\alpha\tau w} \qquad \alpha > 1$$

where,

$$C(0) = K\alpha$$

Since $G_w(0) = 1,\ K\alpha = 9$, for 0.1 steady state error.

Now let us modify the system transfer function by introducing *K* to the original system. Thus the modified system becomes

$$G_m(w) = \frac{-0.00025K(w-20)(w+400)}{(w+1)(w+2)}$$

PM of the closed loop system should be 50°. Let the gain crossover frequency of the uncompensated system with *K* be ω_g . Then,

$$
\begin{aligned}
Mag.(G_m) &= \frac{0.00025K\sqrt{400+\omega^2}\sqrt{160000+\omega^2}}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}} \\
Phase(G_m) &= -\tan^{-1}\omega - \tan^{-1}0.5\omega - \tan^{-1}0.05\omega + \tan^{-1}0.0025\omega
\end{aligned}
$$

Required PM is 50°. Let us put a safety margin of 5°. Thus the PM of the system modified with *K*should be 55°.

$$
\begin{aligned}
\Rightarrow 180^o - \tan^{-1}\omega_g - \tan^{-1}0.5\omega_g - \tan^{-1}0.05\omega_g + \tan^{-1}0.0025\omega_g &= 55^o \\
\text{or, } \tan^{-1}\frac{\omega_g + 0.5\omega_g}{1-0.5\omega_g^2} - \tan^{-1}\frac{0.05\omega_g - 0.0025\omega_g}{1+0.000125\omega_g^2} &= 125^o
\end{aligned}
$$

By solving the above, $\omega_g = 2.44$ rad/sec. Thus the magnitude at ω_g should be 1.

$$
\Rightarrow \frac{0.00025K\sqrt{400+\omega_g^2}\sqrt{160000+\omega_g^2}}{\sqrt{1+\omega_g^2}\sqrt{4+\omega_g^2}} = 1
$$

Putting the value of ω_g in the last equation, we get $K = 4.13$.

Thus,

$$
\alpha = \frac{9}{K} = 2.18
$$

If we place $1/\tau$ one decade below the gain crossover frequency, then

$$
\frac{1}{\tau} = \frac{2.44}{10}, \quad \text{or,} \quad \tau = 4.1
$$

Thus the controller in w -plane is

$$
C(w) = 9\frac{1+4.1w}{1+8.9w}
$$

Re-transforming the above controller into z -plane using the relation $w = 20\frac{z-1}{z+1}$, we get

$$C_z(z) = 9\,\frac{1+20\times 4.1\times\dfrac{z-1}{z+1}}{1+20\times 8.9\times\dfrac{z-1}{z+1}} = 9\,\frac{83z-81}{179z-177}$$

The Bode plot of the compensated system is shown in Figure 4.

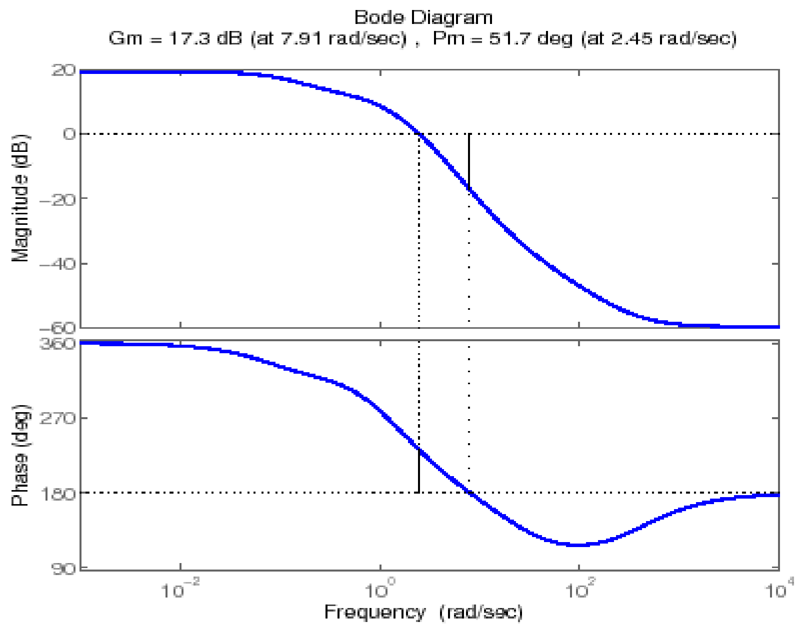


Figure 4: Bode plot of the compensated system for Example 2

➢ Lag -lead Compensator

When a single lead or lag compensator cannot guarantee the specified design criteria, a lag-lead compensator is used.

In lag-lead compensator the lag part precedes the lead part. A continuous time lag-lead compensator is given by

$$C(s) = K\,\frac{1+\tau_1 s}{1+\alpha_1\tau_1 s}\,\frac{1+\tau_2 s}{1+\alpha_2\tau_2 s}$$ where, $\alpha_1 > 1,\ \ \alpha_2 < 1$

The corner frequencies are $\frac{1}{\alpha_1\tau_1}$, $\frac{1}{\tau_1}$, $\frac{1}{\tau_2}$, $\frac{1}{\alpha_2\tau_2}$. The frequency response is shown in Figure 1.

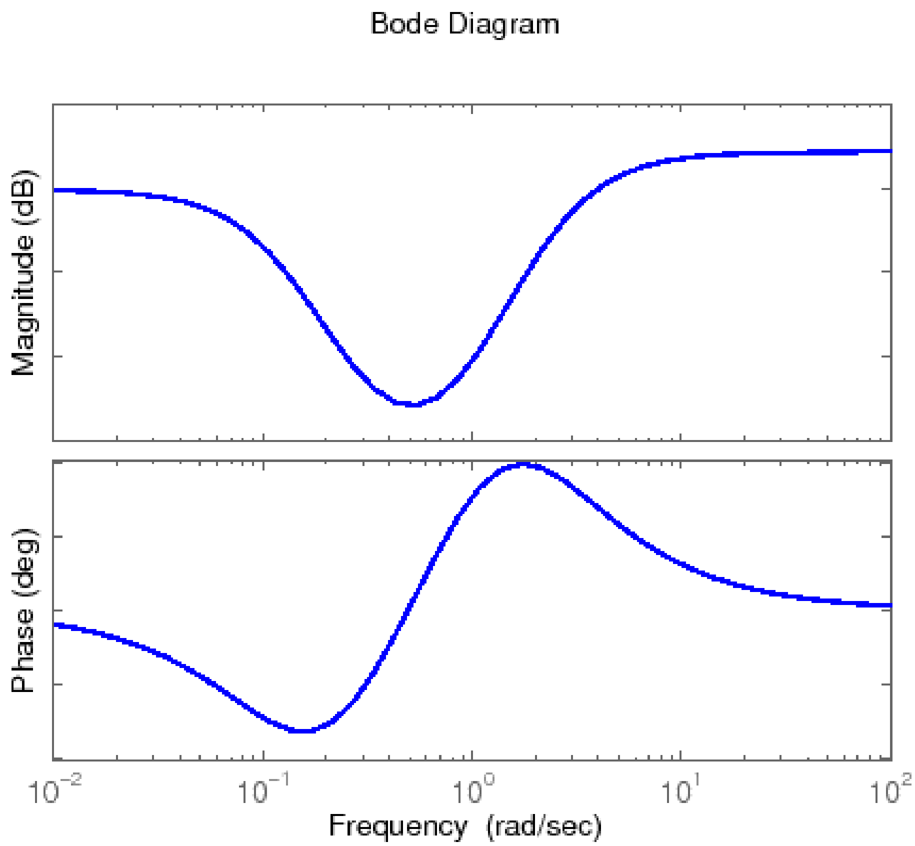


Figure 1: Frequency response of a lag-lead compensator

In a nutshell,

If it is not specified which type of compensator has to be designed, one should first check the PM and BW of the uncompensated system with adjustable gain *K*.

If the BW is smaller than the acceptable BW one may go for lead compensator. If the BW is large, lead compensator may not be useful since it provides high frequency amplification.

One may go for a lag compensator when BW is large provided the open loop system is stable.

If the lag compensator results in a too low BW (slow speed of response), a lag-lead compensator may be used.

- **Lag-lead compensator design**

Example 1 Consider the following system with transfer function

$$G(s) = \frac{1}{s(1+0.1s)(1+0.2s)}$$

Design a lag-lead compensator C(s) such that the phase margin of the compensated system is at least 45° at gain crossover frequency around 10 rad/sec and the velocity error constant K_v is 30.

The lag-lead compensator is given by

$$C(s) = K\,\frac{1+\tau_1 s}{1+\alpha_1\tau_1 s}\,\frac{1+\tau_2 s}{1+\alpha_2\tau_2 s} \quad \text{where,} \quad \alpha_1 > 1,\ \ \alpha_2 < 1$$

When $s \to 0,\ C(s) \to K.$

$$K_v = \lim_{s\to 0} sG(s)C(s) = C(0) = 30$$

Thus $K = 30$. Bode plot of the modified system KG(s) is shown in Figure 2. The gain crossover frequency and phase margin of KG(s) are found out to be 9.77 rad/sec and -17.2° respectively.

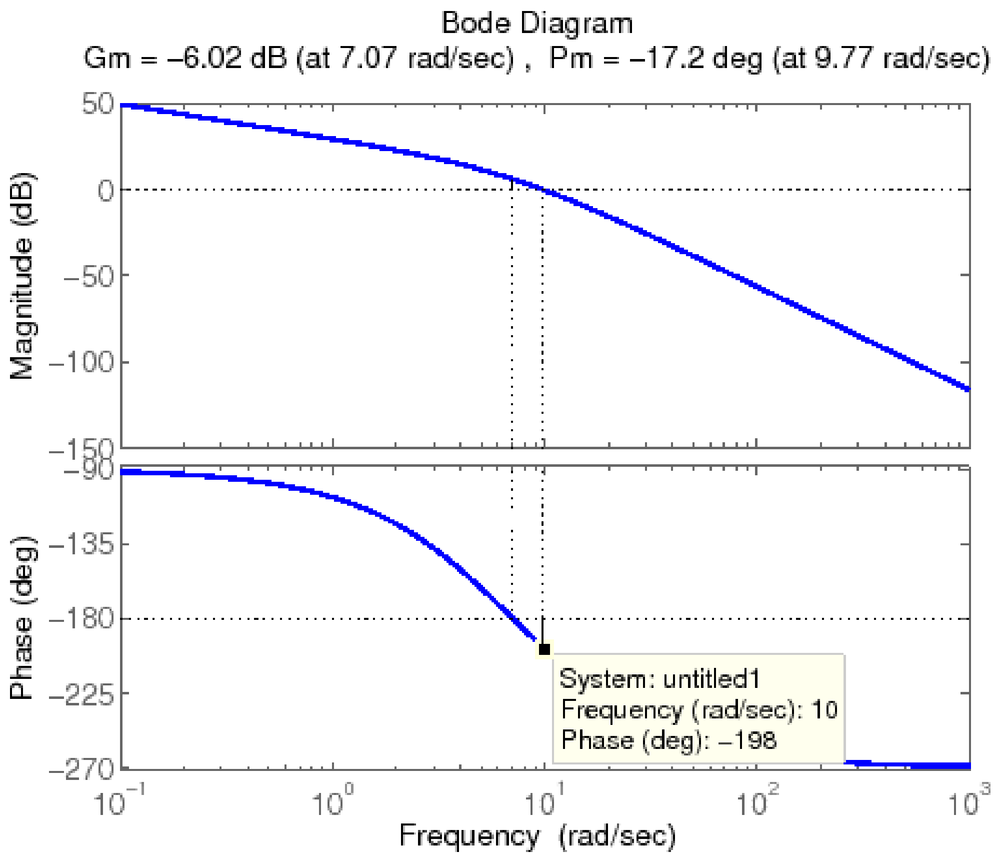


Figure 2: Bode plot of the uncompensated system for Example 1

Since the PM of the uncompensated system with *K* is negative, we need a lead compensator to compensate for the negative PM and achieve the desired phase margin.

However, we know that introduction of a lead compensator will eventually increase the gain crossover frequency to maintain the low frequency gain.

Thus the gain crossover frequency of the system cascaded with a lead compensator is likely to be much above the specified one, since the gain crossover frequency of the uncompensated system with*K* is already 9.77 rad/sec.

Thus a lag-lead compensator is required to compensate for both.

We design the lead part first.

From Figure 2, it is seen that at 10 rad/sec the phase angle of the system is -198°.

Since the new ω_g should be 10 rad/sec, the required additional phase at ω_g, to maintain the specified PM, is 45 - (180 - 198) = 63° . With safety margin 2°,

$$\alpha_2 = \left(\frac{1 - \sin(65^o)}{1 + \sin(65^o)} \right) = 0.05$$

And

$$10 = \frac{1}{\tau_2 \sqrt{\alpha_2}}$$

which gives $\tau_2 = 0.45$. However, introducing this compensator will actually increase the gain crossover frequency where the phase characteristic will be different than the designed one. This can be seen from Figure 3.

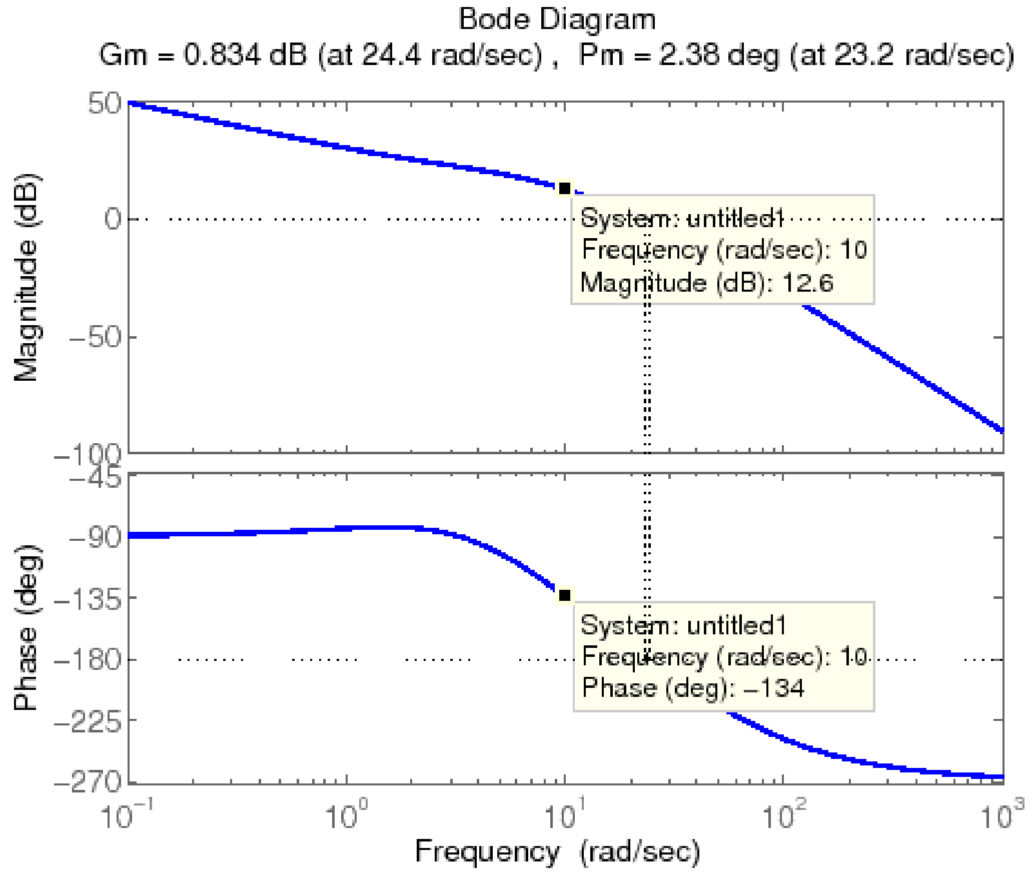


Figure 3: Frequency response of the system in Example 1 with only a lead compensator

The gain crossover frequency is increased to 23.2 rad/sec. At 10 rad/sec, the phase angle is -134° and gain is 12.6 dB. To make this as the actual gain crossover frequency, lag part should provide an attenuation of -12.6 dB at high frequencies.

At high frequencies the magnitude of the lag compensator part is $1/\alpha_1$. Thus ,

$$20\log_{10}\alpha_1 = 12.6$$

which gives $\alpha_1 = 4.27$. Now, $1/\tau_1$ should be placed much below the new gain crossover frequency to retain the desired PM. Let $1/\tau_1$ be 0.25. Thus

$$\tau_1 = 4$$

The overall compensator is

$$C(s) = 30\,\frac{1+4s}{1+17.08s}\,\frac{1+0.45s}{1+0.0225s}$$

The frequency response of the system after introducing the above compensator is shown in Figure 4, which shows that the desired performance criteria are met.

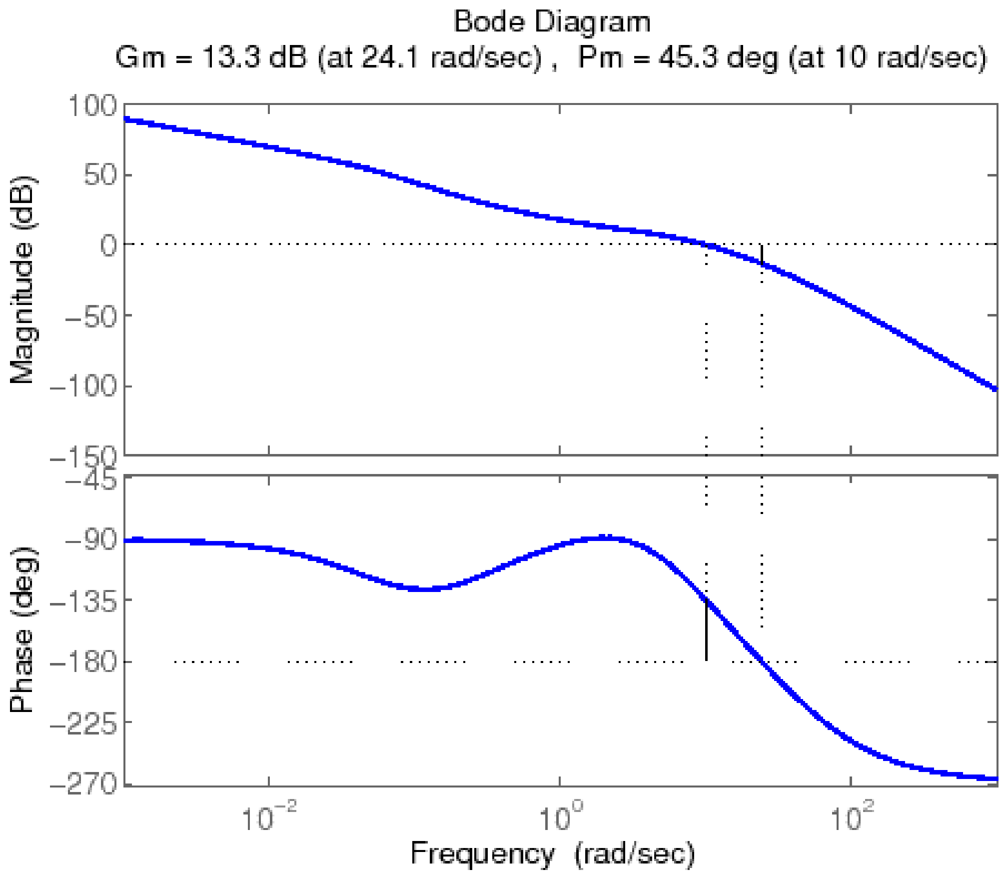


Figure 4: Frequency response of the system in Example 1 with a lag-lead compensator

Example 2:

Now let us consider that the system as described in the previous example is subject to a sampled data control system with sampling time $T = 0.1$ sec. We would use MATLAB to derive the plant transfer function w-plane.

Use the below commands.

```
>> s=tf('s');
>> gc=1/(s*(1+0.1*s)*(1+0.2*s));
>> gz=c2d(gc,0.1,'zoh');
```

You would get

$$G_z(z) = \frac{0.005824z^2 + 0.01629z + 0.002753}{z^3 - 1.974z^2 + 1.198z - 0.2231}$$

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.05w)}{(1 - 0.05w)}$$

will transfer $G_z(z)$ into w-plane. Use the below commands

```
>> aug=[0.1,1];
>> gwss = bilin(ss(gz),-1,'S_Tust',aug)
>> gw=tf(gwss)
```

to find out the transfer function in w-plane, as

$$G_w(w) = \frac{0.001756w^3 - 0.06306w^2 - 1.705w + 45.27}{w^3 + 14.14w^2 + 45.27w - 5.629 \times 10^{-13}}$$

Since the velocity error constant criterion will produce the same controller dcgain K, the gain of the lag-lead compensator is designed to be 30.

The Bode plot of the uncompensated system with $K = 30$ is shown in Figure 5.

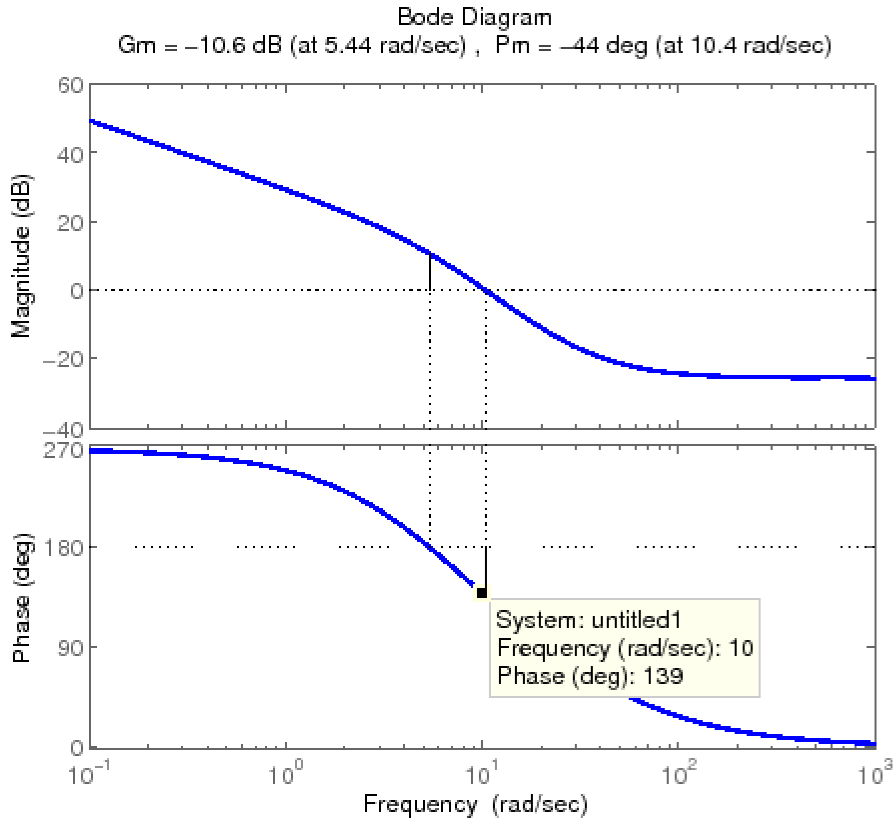


Figure 5: Bode plot of the uncompensated system for Example 2

From Figure 5, it is seen that at 10 rad/sec the phase angle of the system is $139 = -221^o$.

Thus a huge phase lead (86^o) is required if we want to acieve a PM of 45^o which is not possible with a single lead compensator. Let us lower the PM requirement to a minimum of 20^o at $\omega_g = 10$ rad/sec.

Since the new ω_g should be 10 rad/sec, the required additional phase at ω_g, to maintain the specified PM, is $20 - (180 - 221) = 61^o$. With safety margin 5^o,

$$\alpha_2 = \left(\frac{1 - \sin(66^o)}{1 + \sin(66^o)}\right) = 0.045$$

And

$$10 = \frac{1}{\tau_2\sqrt{\alpha_2}}$$

which gives $\tau_2 = 0.47$. However, introducing this compensator will actually increase the gain crossover frequency where the phase characteristic will be different than the designed one. This can be seen from Figure 6.

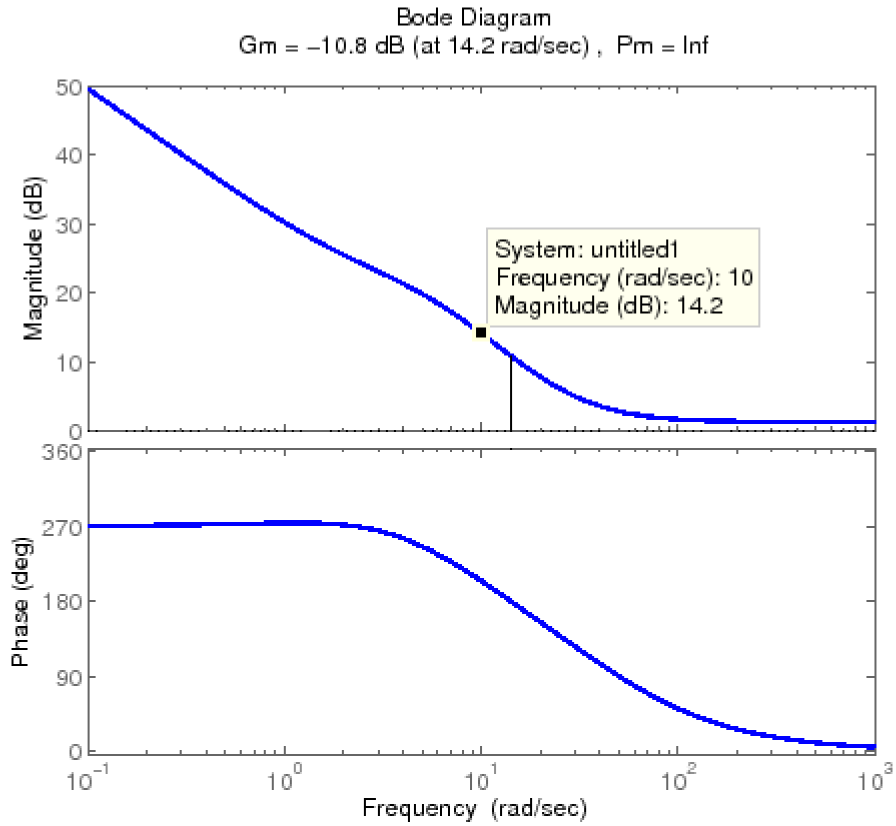


Figure 6: Frequency response of the system in Example 2 with only a lead compensator

Also, as seen from Figure 6, the GM of the system is negative. Thus we need a lag compensator to lower the magnitude at 10 rad/sec.

At 10 rad/sec, the magnitude is 14.2 dB. To make this as the actual gain crossover frequency, lag part should provide an attenuation of -14.2 dB at high frequencies. Thus,

$$20\log_{10}\alpha_1 = 14.2$$

which gives $\alpha_1 = 5.11$. Now, $1/\tau_1$ should be placed much below the new gain crossover frequency to retain the desired PM. Let $1/\tau_1$ be $10/10 = 1$. Thus

$$\tau_1 = 1$$

The overall compensator is

$$C(w) = 30\left(\frac{1+w}{1+5.11w}\right)\left(\frac{1+0.47w}{1+0.02115w}\right)$$

The frequency response of the system after introducing the above compensator is shown in Figure 7, which shows that the desired performance criteria are met.

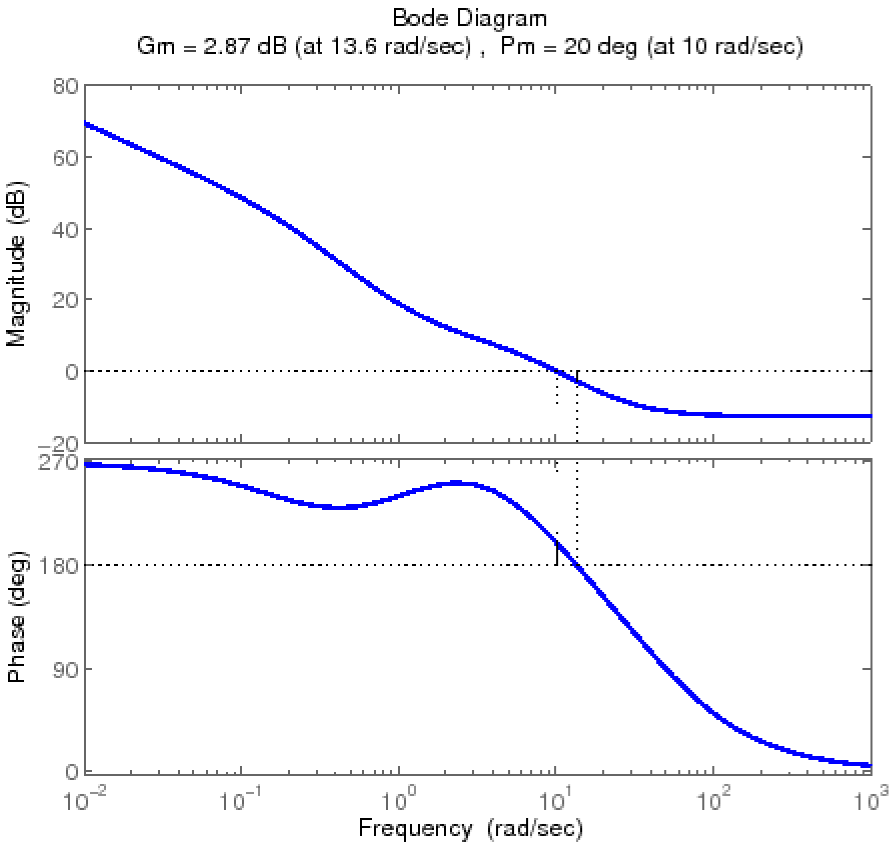


Figure 7: Frequency response of the system in Example 2 with a lag-lead compensator

Re-converting the controller in z-domain, we get

$$C(z) = 30\left(\frac{0.2035z - 0.1841}{z - 0.9806}\right)\left(\frac{7.309z - 5.903}{z + 0.4055}\right)$$

- **Design based on root locus method**

The effect of system gain and/or sampling period on the absolute and relative stability of the closed loop system should be investigated in addition to the transient response characteristics. Root locus method is very useful in this regard.

The root locus method for continuous time systems can be extended to discrete time systems without much modifications since the characteristic equation of a discrete control system is of the same form as that of a continuous time control system.

In many LTI discrete time control systems, the characteristics equation may have either of the following two forms.

$$1 + G(z)H(z) = 0$$

$$1 + GH(z) = 0$$

To combine both, let us define the characteristics equation as:

$$1 + L(z) = 0 \qquad (1)$$

where, $L(z) = G(z)H(z)$ or $L(z) = GH(z)$. $L(z)$ is popularly known as the loop pulse transfer function. From equation (1), we can write

$$L(z) = -1$$

Since $L(z)$ is a complex quantity it can be split into two equations by equating angles and magnitudes of two sides. This gives us the angle and magnitude criteria as

Angle Criterion: $\angle L(z) = \pm 180^0(2k+1), \quad k = 0, 1, 2....$

Magnitude Criterion: $|L(z)| = 1$

The values of *z* that satisfy both criteria are the roots of the characteristics equation or close loop poles. Before constructing the root locus, the characteristics equation $1 + L(z) = 0$ should be rearranged in the following form

$$1 + K\frac{(z+z_1)(z+z_2)....(z+z_m)}{(z+p_1)(z+p_2)....(z+p_n)} = 0$$

where z_i's and p_i's are zeros and poles of open loop transfer function, *m* is the number of zeros *n*is the number of poles.

- **Construction Rules for Root Locus**

Root locus construction rules for digital systems are same as that of continuous time systems.

1. The root locus is symmetric about real axis. Number of root locus branches equals the number of open loop poles.

2. The root locus branches start from the open loop poles at $K = 0$ and ends at open loop zeros at $K = \infty$. In absence of open loop zeros, the locus tends to ∞ when $K \to \infty$. Number of branches that tend to ∞ is equal to difference between the number of poles and number of zeros.

3. A portion of the real axis will be a part of the root locus if the number of poles plus number of zeros to the right of that portion is odd.

4. If there are *n* open loop poles and *m* open loop zeros then $n - m$ root locus branches tend to ∞ along the straight line asymptotes drawn from a single point $s = \sigma$ which is called centroid of the loci.

$$\phi_q = \frac{180^o(2q+1)}{n-m}, \quad q = 0, 1, \ldots, n-m-1$$

Angle of asymptotes

5. Breakaway (Break in) points or the points of multiple roots are the solution of the following equation:

$$\frac{dK}{dz} = 0$$

where *K* is expressed as a function of *z* from the characteristic equation. This is a necessary but not sufficient condition. One has to check if the solutions lie on the root locus.

6. The intersection (if any) of the root locus with the imaginary axis can be determined from the Routh array.

7. The angle of departure from a complex open loop pole is given by

$$\phi_p = 180^o + \phi$$

where is the net angle contribution of all other open loop poles and zeros to that pole.

$$\phi = \sum_i \psi_i - \sum_{j \neq p} \gamma_j$$

ψ_i's are the angles contributed by zeros and γ_j's are the angles contributed by the poles.

8. The angle of arrival at a complex zero is given by

$$\phi_z = 180^o - \phi$$

where ϕ is same as in the above rule.

9. The gain at any point z_0 on the root locus is given by

$$K = \frac{\prod_{j=1}^{n} |z_0 + p_j|}{\prod_{i=1}^{m} |z_0 + z_i|}$$

- **Root locus diagram of digital control systems**

We will first investigate the effect of controller gain *K* and sampling time *T* on the relative stability of the closed loop system as shown in Figure 1.

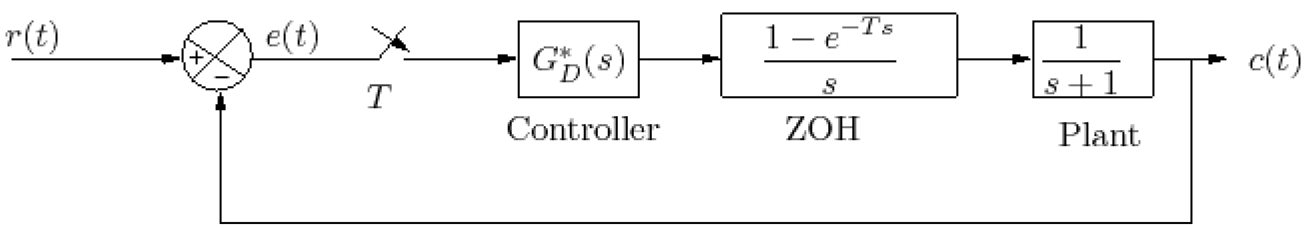


Figure 1: A discrete time control system

Let us first take T=0.5 sec.

$$
\begin{aligned}
Z\left[G_{ho}(s)G_p(s)\right] &= Z\left[\frac{1-e^{-Ts}}{s}.\frac{1}{s+1}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s(s+1)}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s}-\frac{1}{s+1}\right] \\
&= \frac{z-1}{z}\left[\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right] \\
&= \frac{1-e^{-T}}{z-e^{-T}}
\end{aligned}
$$

Let us assume that the controller is an integral controller, i.e., $G_D(z) = \frac{Kz}{z-1}$. Thus,

$$
\begin{aligned}
G(z) &= G_D(z)\cdot G_hG_p(z) \\
&= \frac{Kz}{z-1}\cdot\frac{1-e^{-T}}{z-e^{-T}}
\end{aligned}
$$

The characteristic equation can be written as

$$
\begin{aligned}
& 1+G(z)=0 \\
\Rightarrow \quad & 1+\frac{Kz(1-e^{-T})}{(z-1)(z-e^{-T})}=0 \\
\text{when } T=0.5sec, \quad & L(z)=\frac{0.3935Kz}{(z-1)(z-0.6065)}
\end{aligned}
$$

$L(z)$ has poles at $z = 1$ and $z = 0.605$ and zero at $z = 0$.

Break away/ break in points are calculated by putting $\frac{dK}{dz}=0$.

$$
\begin{aligned}
K &= -\frac{(z-1)(z-0.6065)}{0.3935z} \\
\frac{dK}{dz} &= -\frac{z^2-0.6065}{0.3935z^2}=0 \\
\Rightarrow z^2 = 0.6065 &\Rightarrow z_1 = 0.7788 \text{ and } z_2=-0.7788
\end{aligned}
$$

Critical value of *K* can be found out from the magnitude criterion.

$$\left|\frac{0.3935z}{(z-1)(z-0.6065)}\right| = \frac{1}{K}$$

Critical gain corresponds to point $z = -1$. Thus

$$\left|\frac{-0.03935}{(-2)(-1.6065)}\right| = \frac{1}{K}$$
$$\text{or,} \quad K = 8.165$$

Figure 2 shows the root locus of the system for *K* = 0 to *K* =10. Two root locus branches start from two open loop poles at *K* = 0. If we further increase *K* one branch will go towards the zero and the other one will tend to infinity. The blue circle represents the unit circle. Thus the stable range of *K* is0 < *K* < 8.165.

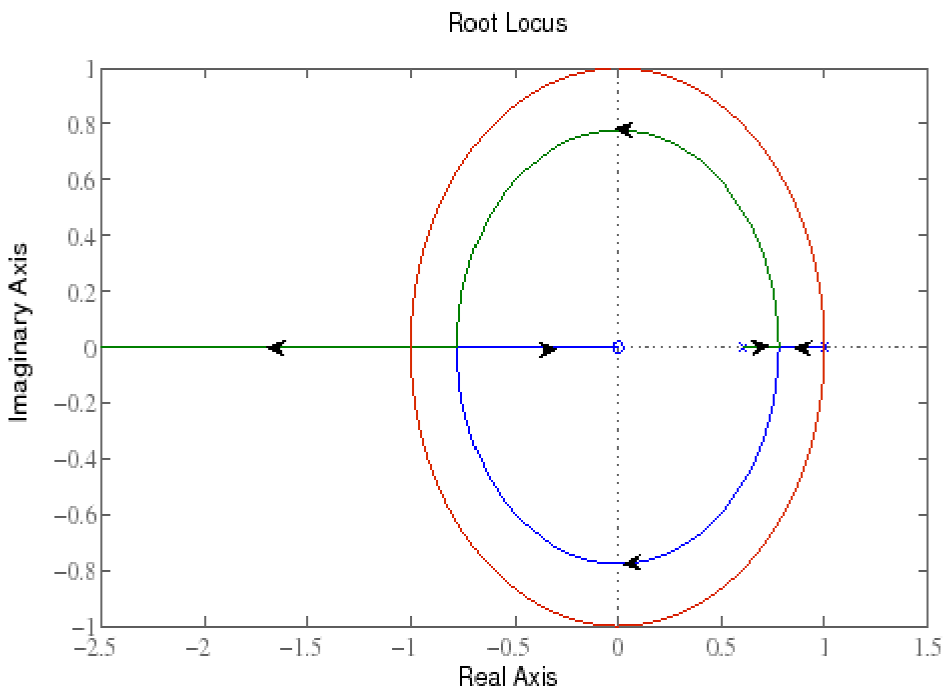


Figure 2: Root Locus when T=0.5 sec

If *T* = 1 sec,

$$G(z) = \frac{0.6321Kz}{(z-1)(z-0.3679)}$$

Break away/ break in points: $z^2 = 0.3679 \Rightarrow z_1 = 0.6065$ and $z_2 = -0.6065$ Critical gain $(K_c) = 4.328$ Figure 3 shows the root locus for K = 0 to K = 10. It can be seen from the figure that the3 radius of the inside circle reduces and the maximum value of stable K also decreases to K = 4.328 .

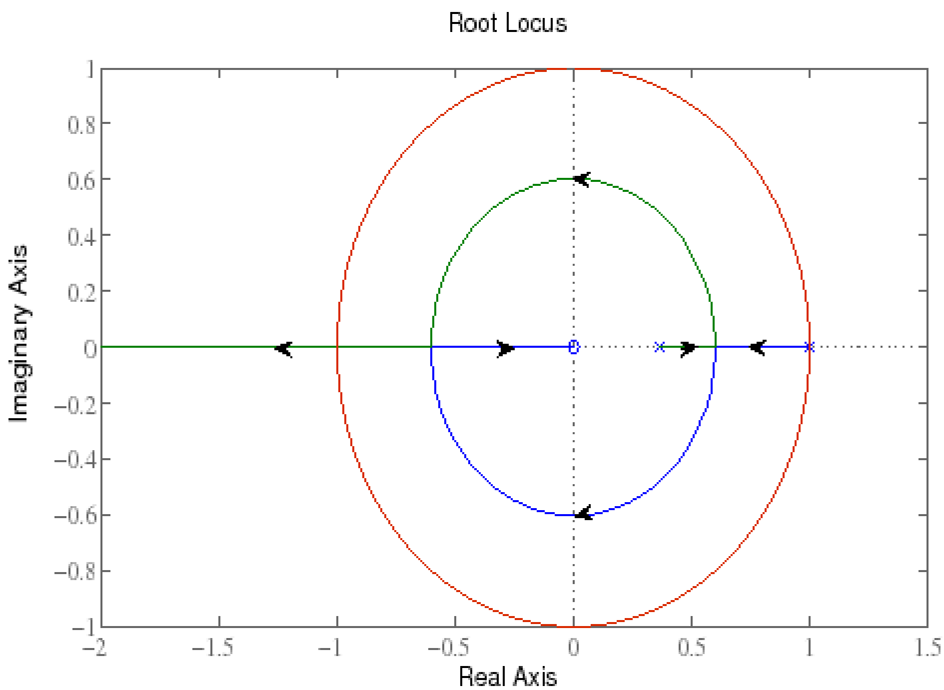


Figure 3: Root Locus when T=1 sec

Similarly if T = 2 sec,

$$G(z) = \frac{0.8647Kz}{(z-1)(z-0.1353)}$$

One can find that the critical gain in this case further reduces to 2.626.

- **Effect of sampling period T**

As can be seen from the previous example, large T has detrimental effect on relative stability. A thumb rule is to sample eight to ten times during a cycle of the damped sinusoidal oscillation of the output if it is underdamped. If overdamped, 8/10 times during rise time.

As seen from the example making the sampling period smaller allows the critical gain to be larger, i.e., maximum allowable gain can be made larger by increasing sampling frequency /rate.

It seems from the example that damping ratio decreases with the decrease in *T*. But one should take a note that damping ratio of the closed loop poles of a digital control system indicates the relative stability only if the sampling frequency is sufficiently high (8 to 10 times). If it is not the case, prediction of overshoot from the damping ratio will be erroneous and in practice the overshoot will be much higher than the predicted one.

Next, we may investigate the effect of *T* on the steady state error. Let us take a fixed gain *K* = 2.

When *T* = 0.5 sec. and *K* = 2,

$$G(z) = \frac{0.787z}{(z-1)(z-0.6065)}$$

Since this is a second order system, velocity error constant will be a non zero finite quantity.

$$Kv = \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} = 4$$

$$e_{ss} = \frac{1}{4} = 0.25$$

Thus,

When *T* = 1 sec. and *K* = 2

$$G(z) = \frac{1.2642z}{(z-1)(z-0.3679)}$$

$$Kv = \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} = 2$$

$$e_{ss} = \frac{1}{2} = 0.5$$

When *T* = 2 sec. and *K* = 2

$$G(z) = \frac{1.7294z}{(z-1)(z-0.1353)}$$

$$Kv = \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} = 1$$

$$e_{ss} = \frac{1}{1} = 1$$

Thus, increasing sampling period (decreasing sampling frequency) has an adverse effect on the steady state error as well.

- **Design by using Root Locus**
  - **Controller types:** We have already studied different variants of controllers such as PI, PD, PID etc. We know that PI controller is generally used to improve steady state performance whereas PD controller is used to improve the relative stability or transient response. Similarly a phase lead compensator improves the dynamic performance whereas a lag compensator improves the steady state response.

  - **Pole-Zero cancellation** A common practice in designing controllers in s-plane or z-plane is to cancel the undesired poles or zeros of plant transfer function by the zeros and poles of controller. New poles and zeros can also be added in some advantageous locations. However, one has to keep in mind that pole-zero cancellation scheme does not always provide satisfactory solution. Moreover, if the undesired poles are near $j\omega$ axis, inexact cancellation, which is almost inevitable in practice, may lead to a marginally stable or even unstable closed loop system. For this reason one should never try to cancel an unstable pole.

- **Design Procedure:** Consider a compensator of the form $K\frac{z+a}{z+b}$. It will be a lead compensator if the zero lies on the right of the pole.

1. Calculate the desired closed loop pole pairs based on design criteria.

2. Map the s-domain poles to z-domain.

3. Check if the sampling frequency is 8 - 10 times the desired damped frequency of oscillation.

4. Calculate the angle contributions of all open loop poles and zeros to the desired closed loop pole.

5. Compute the required contribution by the controller transfer function to satisfy angle criterion.

6. Place the controller zero in a suitable location and calculate the required angle contribution of the controller pole.

7. Compute the location of the controller pole to provide the required angle.

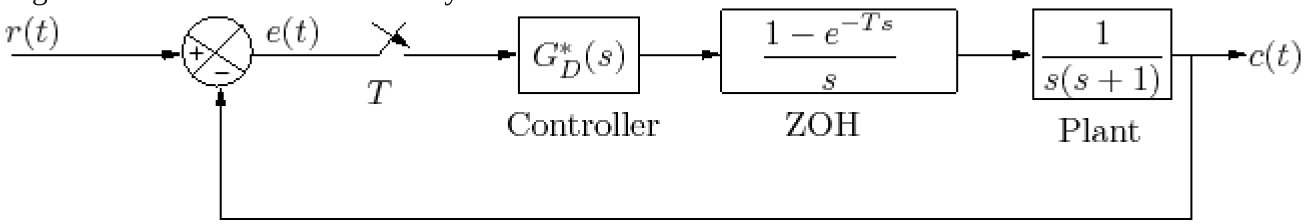
8. Find out the gain K from the magnitude criterion.

The following example will illustrate the design procedure.

- **An Example on Controller Design**

  Consider the closed loop discrete control system as shown in Figure 1 .

Figure 1: A discrete time control system



Design a digital controller such that the dominant closed loop poles have a damping ratio $\xi = 0.5$ and settling time $t_s = 2$ sec for 2% tolerance band. Take the sampling period as T = 0.2 sec. The dominant pole pair in continuous domain is $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ where ω_n is the natural undamped frequency.

$$\begin{aligned}
\text{Given that settling time } t_s &= \frac{4}{\xi\omega_n} = \frac{4}{0.5\omega_n} = 2 \text{ sec.} \\
\text{Thus, } \omega_n &= 4 \\
\text{Damped frequency } \omega_d &= 4\sqrt{1-0.5^2} = 3.46 \\
\text{Sampling frequency } \omega_s &= \frac{2\pi}{T} = \frac{2\pi}{0.2} = 31.4
\end{aligned}$$

Since $\frac{31.4}{3.46} = 9.07$, we get approximately 9 samples per cycle of the damped oscillation. The closed loop poles in s-plane

$$\begin{aligned} s_{1,2} &= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \\ &= -2 \pm j3.46 \end{aligned}$$

Thus the closed loop poles in z-plane

$$\begin{aligned} z_{1,2} &= exp(T(-2 \pm j3.46)) \\ |z| &= e^{-T\xi\omega_n} = exp(-0.4) = 0.67 \\ \angle z &= T\omega_d = 0.2 \times 3.464 = 0.69 \text{ rad} = 39.69^0 \\ \text{Thus, } z_{1,2} &= 0.67\angle 39.7^0 \cong 0.52 \pm j0.43 \end{aligned}$$

$$\begin{aligned} G(z) &= Z\left[\frac{1-e^{-0.2s}}{s}.\frac{1}{s(s+1)}\right] \\ &= (1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right] \\ &\cong \frac{0.02(z+0.93)}{(z-1)(z-0.82)} \end{aligned}$$

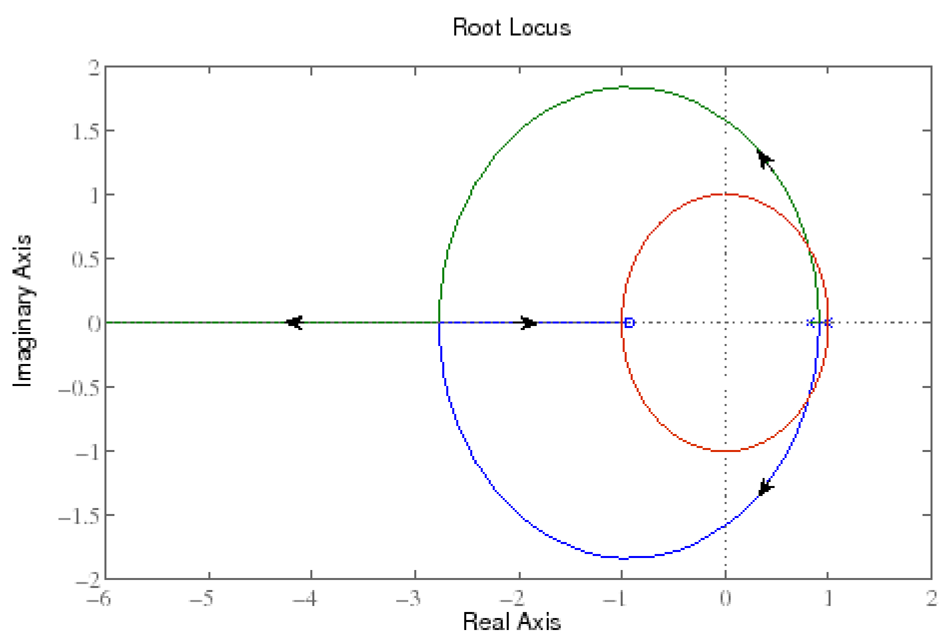


Figure 2: Root locus of uncompensated system

The root locus of the uncompensated system (without controller) is shown in Figure 2. It is clear from the root locus plot that the uncompensated system is stable for a very small range of *K*.

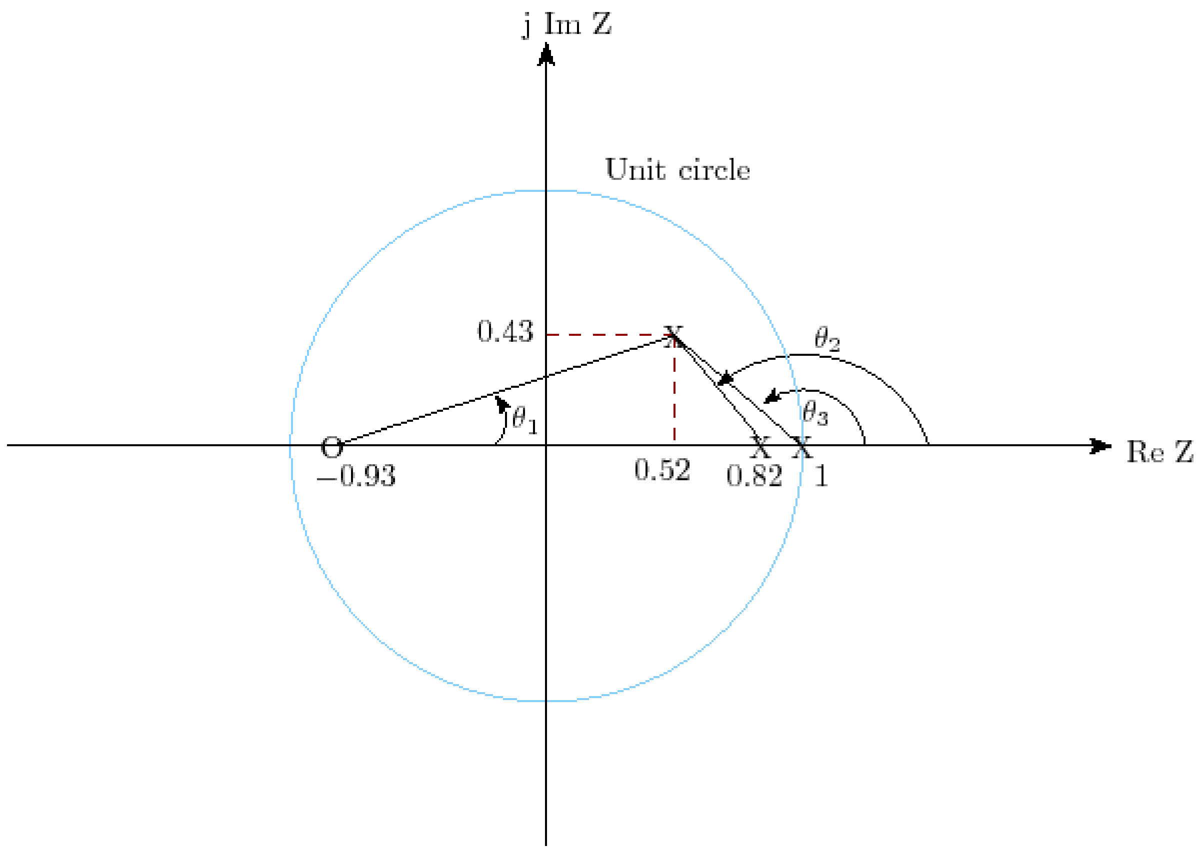


Figure 3: Pole zero map to compute angle contributions

Pole zero map of the uncompensated system is shown in Figure 3. Sum of angle contributions at the desired pole is $A = \theta_1 - \theta_2 - \theta_3$, where θ_1 is the angle by the zero, -0.93, and θ_2 and θ_3 are the angles contributed by the two poles, 0.82 and 1 respectively.

From the pole zero map as shown in Figure 3, the angles can be calculated as $\theta_1 = 16.5^o$, $\theta_2 = 124.9^o$ and $\theta_3 = 138.1^o$.

Net angle contribution is $A = 16.5^o - 124.9^o - 138.1^o = -246.5^o$. But from angle criterion a point will lie on root locus if the total angle contribution at that point is $\pm 180^o$. Angle deficiency is $-246.5^o + 180^o = -66.5^o$

Controller pulse transfer function must provide an angle of 66.5°. Thus we need a Lead Compensator. Let us consider the following compensator.

$$G_D(z) = K\frac{z+a}{z+b}$$

If we place controller zero at $z = 0.82$ to cancel the pole there, we can avoid some of the calculations involved in the design. Then the controller pole should provide an angle of $124.9^o - 66.5^o = 58.4^o$.

Once we know the required angle contribution of the controller pole, we can easily calculate the pole location as follows.

The pole location is already assumed at $z = -b$. Since the required angle is greater than $\tan^{-1}(0.43/0.52) = 39.6^o$ we can easily say that the pole must lie on the right half of the unit circle. Thus *b* should be negative. To satisfy angle criterion,

$$
\begin{aligned}
\tan^{-1}\frac{0.43}{0.52-|b|} &= 58.4^o \\
\text{or, } \frac{0.43}{0.52-|b|} &= \tan(58.4^o) = 1.625 \\
\text{or, } 0.52-|b| &= \frac{0.43}{1.56} = 0.267 \\
\text{or, } |b| &= 0.52 - 0.267 = 0.253 \\
\text{Thus, } b &= -0.253
\end{aligned}
$$

The controller is then written as $G_D(z) = K\frac{z-0.82}{z-0.253}$. The root locus of the compensated system (with controller) is shown in Figure 4.

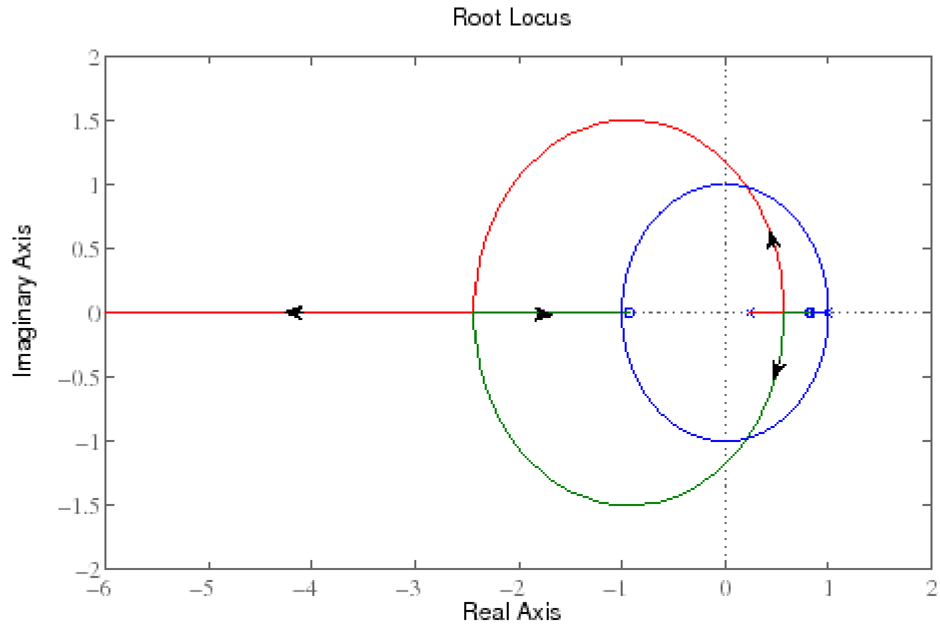


Figure 4: Root locus of the compensated system

If we compare Figure 4 with Figure 2, it is evident that stable region of *K* is much larger for the compensated system than the uncompensated system. Next we need to calculate *K* from the magnitude criterion.

$$\text{Magnitude criterion:} \quad \left|\frac{0.02K(z+0.93)}{(z-0.253)(z-1)}\right|_{z=0.52+j0.43} = 1$$

$$\text{or,} \quad K = \left|\frac{(z-0.253)(z-1)}{0.02(z+0.93)}\right|_{z=0.52+j0.43}$$

$$= \frac{|0.52+j0.43-0.253||0.52+j0.43-1|}{0.02|0.52+j0.43+0.93|} = 10.75$$

Thus the required controller is $G_D(z) = 10.75\dfrac{z-0.82}{z-0.253}$. The SIMULINK block to compute the output response is shown in Figure 5. All discrete blocks in the SIMULINK model should have same sampling period which is 0.2 sec in this example.

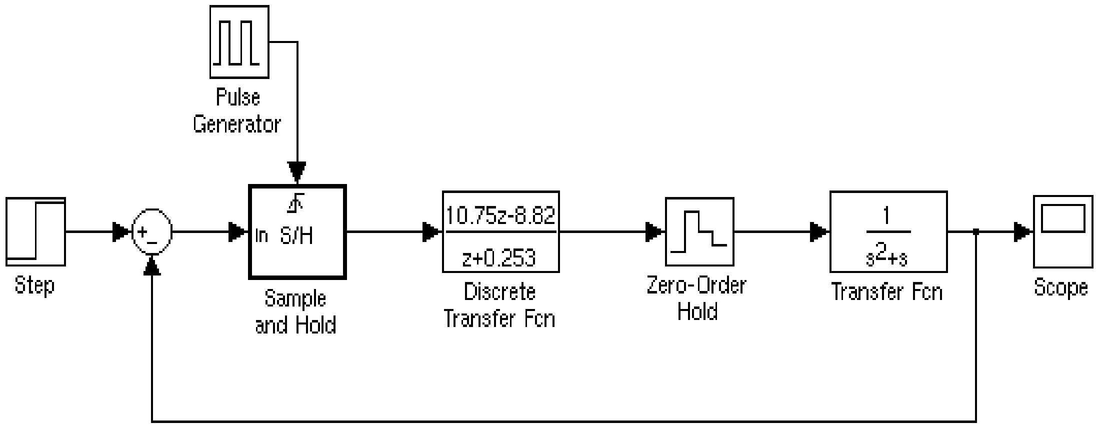


Figure 5: Simulink diagram of the closed loop system

Unit – VI

State Feedback Controllers

UNIT SYLLABUS

Design of state feedback controller through pole placement – Necessary and sufficient conditions – Ackerman's formula.

4.1.1. Unit Objectives:

After reading this Unit, you should be able to understand:

- To study the design of state feedback control by "the pole placement method."

4.1.2. Unit Outcomes:

- The learner understand the stability of digital control systems and how to make the unstable system to stable. The leaner will understand about designing of systems by conventional methods like root locus and bode plot through bilinear transformation.

The design techniques described in the preceding lectures are based on the transfer function of a system. In this lecture we would discuss the state variable methods of designing controllers. The advantageous of state variable method will be apparent when we design controllers for multi input multi output systems. Moreover, transfer function methods are applicable only for linear time invariant and initially relaxed systems.

➢ **State Feedback Controller**

Consider the state-space model of a SISO system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) \end{aligned} \qquad (1)$$

where $\mathbf{x}(k) \in R^n$, *u(k)* and *y(k)* are scalar. In state feedback design, the states are fedback to the input side to place the closed poles at desired locations.

Regulation Problem: When we want the states to approach zero starting from any arbitrary initial state, the design problem is known as regulation where the internal stability of the system, with desired transients, is achieved. Control input:

$$u(k) = -K\mathbf{x}(k) \qquad (2)$$

Tracking Problem: When the output has to track a reference signal, the design problem is known as tracking problem. Control input:

$$u(k) = -K\mathbf{x}(k) + Nr(k)$$

where *r(k)* is the reference signal.

First we will discuss designing a state feedback control law using pole placement technique for regulation problem.

By substituting the control law (2) in the system state model (1), the closed loop system becomes $\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k)$. If *K* can be designed such that eigenvalues of *A-BK* are within the unit circle then the problem of regulation will be solved.

The control problem can thus be defined as: Design a state feedback gain matrix *K* such that the control law given by equation (2) places poles of the closed loop system $\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k)$ in desired locations.

A necessary and sufficient condition for arbitrary pole placement is that the pair (A,B) must be controllable. Since the states are feedback to the input side, we assume that all the states are measurable.

➢ Designing *K* by transforming the state model into controllable from (pole placement technique)

The problem is first solved for the controllable canonical form. Let us denote the controllability matrix by U_C and consider a transformation matrix *T* as

$$T = U_C W$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

a_i's are the coefficients of the characteristic polynomial

$$|zI - A| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

.

Define a new state vector $\mathbf{x} = T\bar{\mathbf{x}}$. This will transform the system given by (1) into controllable canonical form, as

$$\bar{\mathbf{x}}(k+1) = \bar{A}\bar{\mathbf{x}}(k) + \bar{B}u(k) \qquad (3)$$

You should verify that

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \text{ and } \bar{B} = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We first find $\bar{K}$ such that $u(k) = -\bar{K}\bar{\mathbf{x}}(k)$ places poles in desired locations. Since eigenvalues remain unaffected under similarity transformation, $u(k) = -\bar{K}T^{-1}\mathbf{x}(k)$ will also place the poles of the original system in desired locations.

If poles are placed at $z_1, z_2, ..., z_n$, the desired characteristic equation can be expressed as:

$$\begin{aligned} (z - z_1)(z - z_2)\dots(z - z_n) &= 0 \\ \text{or,} \quad z^n + \alpha_1 z^{n-1} + \dots + \alpha_{n-1} z + \alpha_n &= 0 \end{aligned} \tag{4}$$

Since the pair $(\bar{A}, \bar{B})$ are in controllable-companion form then, we have

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -(a_n - \bar{k}_1) & -(a_{n-1} - \bar{k}_2) & & \dots & -(a_1 - \bar{k}_n) \end{bmatrix}$$

Please note that the characteristic equation of both original and canonical form is expressed as: $|zI - A|$ = $|zI - \bar{A}|$ = $z^n + a_1 z^{n-1} + \dots + a_n$ = 0.

The characteristic equation of the closed loop system with $u = -\bar{K}\bar{\mathbf{x}}$ is given as:

$$z^n + (a_1 + \bar{k}_n)z^{n-1} + (a_2 + \bar{k}_{n-1})z^{n-2} + \dots + (a_n + \bar{k}_1) = 0 \tag{5}$$

Comparing Eqs. (4) and (5), we get

$$\bar{k}_n = \alpha_1 - a_1,\ \bar{k}_{n-1} = \alpha_2 - a_2,\ \bar{k}_1 = \alpha_n - a_n \tag{6}$$

We need to compute the transformation matrix *T* to find the actual gain matrix

$K = \bar{K}T^{-1}$ where $\bar{K} = [\bar{k}_1, \bar{k}_2, \dots, \bar{k}_n]$.

- **Designing *K* by Axkermann's Formula**

Consider the state-space model of a SISO system given by equation (1). The control input is

$$u(k) = -K\mathbf{x}(k) \tag{7}$$

Thus the closed loop system will be

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k) = \hat{A}\mathbf{x}(k) \tag{8}$$

Desired characteristic Equation:

$$\begin{aligned} |zI - A + BK| = |zI - \hat{A}| &= 0 \\ \text{or,} \quad (z - z_1)(z - z_2)\cdots(z - z_n) &= 0 \\ \text{or,} \quad z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n &= 0 \end{aligned}$$

$$\hat{A}^n + \alpha_1 \hat{A}^{n-1} + \ldots + \alpha_{n-1}\hat{A} + \alpha_n I = 0$$

Using Cayley-Hamilton Theorem

Consider the case when $n = 3$.

$$\begin{aligned}
\hat{A} &= A - BK \\
\hat{A}^2 = (A-BK)^2 = A^2 - ABK - BKA - BKBK &= A^2 - ABK - BK\hat{A} \\
\hat{A}^3 = (A-BK)^3 &= A^3 - A^2BK - ABK\hat{A} - BK\hat{A}^2
\end{aligned}$$

We can then write

$$\begin{aligned}
\alpha_3 I + \alpha_2\hat{A} + \alpha_1\hat{A}^2 + \hat{A}^3 &= 0 \\
\text{or,}\quad \alpha_3 I + \alpha_2(A-BK) + \alpha_1(A^2 - ABK - BK\hat{A}) + A^3 - A^2BK - ABK\hat{A} - BK\hat{A}^2 &= 0 \\
\text{or, } \alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 - \alpha_2 BK - \alpha_1 ABK - \alpha_1 BK\hat{A} - A^2BK - ABK\hat{A} - BK\hat{A}^2 &= 0
\end{aligned}$$

Thus

$$\begin{aligned}
\phi(A) &= B(\alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2) + AB(\alpha_1 K + K\hat{A}) + A^2BK \\
&= [B \quad AB \quad A^2B] \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
&= U_C \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix}
\end{aligned}$$

where $\phi(.)$ is the closed loop characteristic polynomial and U_C is the controllability matrix. Since U_C is nonsingular

$$\begin{aligned}
U_C^{-1}\phi(A) &= \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
\text{or, } [0 \ 0 \ 1]\, U_C^{-1}\phi(A) &= [0 \ 0 \ 1] \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
\text{or, } K &= [0 \ 0 \ 1]\, U_C^{-1}\phi(A)
\end{aligned}$$

Extending the above for any n,

$K = [0 \ 0 \ \ldots \ 1]\, U_C^{-1}\phi(A)$ where $U_C = [B \ \ AB \ \ A^2B \ \ \ldots \ \ A^{n-1}B]$

The above equation is popularly known as Ackermann's formula

Example 1: Find out the state feedback gain matrix ***K*** for the following system using two different methods such that the closed loop poles are located at 0.5 , 0.6 and 0.7.

$$\mathbf{x}(k+1)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x}(k)+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(k)$$

Solution:

$$U_C=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

The above matrix has rank 3, so the system is controllable.

Open loop characteristic equation:

$$z^3+3z^2+2z+1=0$$

or,

Desired characteristic equation:

$$\begin{aligned} (z-0.5)(z-0.6)(z-0.7) &= 0 \\ \text{or,}\ \ z^3-1.8z^2+1.07z-0.21 &= 0 \end{aligned}$$

Since the open loop system is already in controllable canonical form, *T* =1.

$$K=[\alpha_3-a_3\ \ \alpha_2-a_2\ \ \alpha_1-a_1]$$

where, $\alpha_3=-0.21,\ \alpha_2=1.07,\ \alpha_1=-1.8$ and $a_3=1,\ a_2=2,\ a_1=3$. Thus

$$K=[-1.21\quad -0.93\quad -4.8]$$

Using Ackermann's formula:

$$U_C^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -3 & -1 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \phi(A) &= A^3 - 1.8A^2 + 1.07A - 0.21I \\ &= \begin{bmatrix} -1 & -2 & -3 \\ 3 & 5 & 7 \\ -7 & -11 & -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1.8 \\ 1.8 & 3.6 & 5.4 \\ -5.4 & -9 & -10.6 \end{bmatrix} + \begin{bmatrix} 0 & 1.07 & 0 \\ 0 & 0 & 1.07 \\ -1.07 & -2.14 & -3.21 \end{bmatrix} \\ &\quad + \begin{bmatrix} -0.21 & 0 & 0 \\ 0 & -0.21 & 0 \\ 0 & 0 & -0.21 \end{bmatrix} \\ &= \begin{bmatrix} -1.21 & -0.93 & -4.8 \\ 4.8 & 8.39 & 13.47 \\ -13.47 & -22.14 & -30.02 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} K &= [0\ \ 0\ \ 1]U_C^{-1}\phi(A) \\ &= [0\ \ 0\ \ 1]\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\phi(A) \\ &= [1\ \ 0\ \ 0]\phi(A) = [-1.21\ \ -0.93\ \ -4.8] \end{aligned}$$

Example 2: Find out the state feedback gain matrix *K* for the following system by converting the system into controllable canonical form such that the closed loop poles are located at 0.5 and 0.6.

$$\mathbf{x}(k+1) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k)$$

Solution:

$$U_C = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

The above matrix has rank 2, so the system is controllable.

Open loop characteristic equation:

$$z^2 + 3z + 2 = 0$$

or,

Desired characteristic equation:

$$\begin{aligned} (z - 0.5)(z - 0.6) &= 0 \\ \text{or, } z^2 - 1.1z^2 + 0.3 &= 0 \end{aligned}$$

$$W = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

To convert into controllable canonical form:

$$T = U_C W = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

The transformation matrix:

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Check:

$$\alpha_1 = -1.1, \qquad \alpha_2 = 0.3, \qquad a_1 = 3, \qquad a_2 = 2$$

Now,

$$\bar{K} = [\alpha_2 - a_2 \qquad \alpha_1 - a_1] = [-1.7 \qquad -4.1]$$

Thus

$$K = \bar{K}T^{-1} = [-1.7 \qquad -4.1] \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = [-2.4 \qquad -4.1]$$

We can then write

For all examples and exercise problems the system is considered as

$x(k+1) = Fx(k) + Gu(k)$ ……………………………………………………………..……State Equation

$y(k) = Cx(k) + Du(k)$ …………………………………………………………………...Output Equation

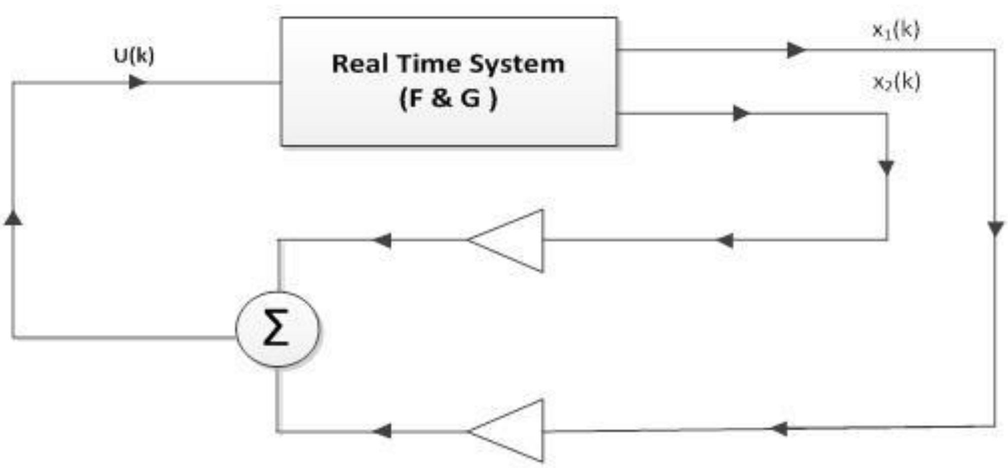
Examples

Example: 1 Given that $F = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Design a state feedback controller such that the closed loop poles are located at z=0.4 & z=0.6

Solution:

The block diagram is given by



From the adjoining block diagram we have

$$u(k) = -\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}^T \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = -k_1 x_1(k) - k_2 x_2(k)$$

The characteristic equation of the poles located at 0.4 and 0.6 is

$$(z-0.4)(z-0.6) = 0$$
$$z^2 - z + 0.24 = 0$$

From the standard characteristic equation of closed loop poles, we have

$Q_d(z) = |zI - (F - Gk)| = 0$

$$\Rightarrow \left| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}^T \right] \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} k_1 & k_2 \\ k_1 & k_2 \end{pmatrix} \right] \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \left[\begin{pmatrix} 1-k_1 & -1-k_2 \\ -k_1 & 1-k_2 \end{pmatrix} \right] \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} z-1+k_1 & 1+k_2 \\ k_1 & z-1+k_2 \end{pmatrix} \right| = 0$$

$$\Rightarrow [(z-1+k_1)*(z-1+k_2)] - [k_1(1+k_2)] = 0$$
$$\Rightarrow (z-1)^2 + k_1(z-1) + k_2(z-1) + k_1k_2 - k_1 - k_1k_2 = 0$$
$$\Rightarrow z^2 - 2z + 1 + (k_1 + k_2)z - k_2 - 2k_1 = 0$$
$$\Rightarrow z^2 + (k_1 + k_2 - 2)z + (-2k_1 - k_2 + 1) = 0$$

On comparison with standard equation, we get

$$\Rightarrow k_1 + k_2 - 2 = -1$$
$$\Rightarrow k_1 + k_2 = 1$$
$$\Rightarrow k_1 = 1 - k_2$$
$$-2k_1 - k_2 + 1 = 0.24$$

$$-2(1-k_2) - k_2 = -0.76$$
$$-2 + 2k_2 - k_2 = -0.76$$
$$k_2 = 1.24$$

$$k_1 = 1 - k_2$$

$$k_1 = 1 - 1.24 = -0.24$$

k= [-0.24 1.24]

These values of the system gain represent the two controller parameters which are feedback to reference input. Finally the matrix form of the state feedback gain of the controller is given above respectfully.

Example 3: Given that

$$F = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \; G = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Determine the state feedback gain by using Ackermann's Formula.

Solution:

$$F = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \; G = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Here the order (n) of F matrix is n=2

c = [G FG]

$$c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$c^{-1} = \frac{Adjn(c)}{|c|} = \frac{[cofact(c)]^T}{|c|} = \frac{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^T}{1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$Q_d(\overline{F}) = \overline{F}^n + c_n \overline{F}^{n-1} + c_{n-1} \overline{F}^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots + c_2 \overline{F} + c_1 I = 0$$

From the characteristic equation we get

$$Q_d(z) = z^2 - z + 0.24I$$

Substitute $z = F$ we have

$$Q_d(F) = F^2 - F + 0.24I$$

$$F^2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$Q_d(F) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0.24 & 0 \\ 0 & 0.24 \end{pmatrix}$$

$$Q_d(F) = \begin{pmatrix} 0.24 & -1 \\ 0 & 0.24 \end{pmatrix}$$

By Ackermann's formula state feedback gain is given by

$$k = [0 \ldots\ldots\ldots\ldots 1] c^{-1} Q_d(F)$$

$k = [0\ 1] \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0.24 & -1 \\ 0 & 0.24 \end{pmatrix}$

$k = [-1\ 1] \begin{pmatrix} 0.24 & -1 \\ 0 & 0.24 \end{pmatrix}$

$k = [-0.24\ 1.24]$

Problem 1: Given that

$F = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Design a state feedback controller such that the closed loop poles are located at $z = 0.6 \pm 0.4j$.

Problem 2: The *F* and *G* matrices are given below

$F = \begin{pmatrix} -0.2 & 0.6 \\ 0.5 & -0.1 \end{pmatrix} \quad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Design a state feedback controller using the transformation matrix that transforms the system into controllable canonical form in such a way that closed loop poles are located at $z = 0.2$ & $z = 0.8$.

Problem 3: Following *F* and *G* matrices of order 2 are given by

$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Determine the state feedback gain by using Ackermann's Formula.

Code No: **R1642021** **R16** **Set No. 1**

IV B.Tech II Semester Regular Examinations, September - 2020

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any FOUR questions from Part-B

*****

PART–A *(14 Marks)*

1. a) What is meant by impulse sampler? [2]
   b) What is the z-transform of sin ωt? [2]
   c) Explain the concept of controllability. [2]
   d) Write comment on the stability of $F(z) = z^2 - 0.25 = 0$ by using Jury's stability criterion? [3]
   e) List out the transient response specifications. [2]
   f) Write statement on necessary condition for design of state feedback controller through pole placement? [3]

PART–B *(4x14 = 56 Marks)*

2. a) List out the applications where DCS are used? Explain any one of them in detail. [7]
   b) Explain the frequency domain characteristics of zero order hold with neat schematic. [7]

3. a) The input-output of a sampled data system is described by the difference equation $y(k+2) + 3y(k+1) + 4y(k) = r(k+1) - r(k)$; $y(0) = y(1) = 0$ $r(0) = 0$, Determine pulse transfer function. Also obtain the unit pulse response of the system. [7]
   b) Find the inverse z-transform of $F(z) = \frac{z(z+1)}{(z-1)(z^2-z+1)}$ by using partial fraction expansion method. [7]

4. a) Obtain the inverse of the matrix $(ZI - G)$ $where\ G = \begin{pmatrix} 0.1 & 0.1 & 0 \\ 0.3 & -0.1 & -0.2 \\ 0 & 0 & -0.3 \end{pmatrix}$ also obtain G^k. [7]
   b) Consider the following system
   $$\frac{y(z)}{u(z)} = \frac{z+1}{z^2 + 1.3z + 0.4}$$
   Obtain (i) Controllable canonical form (ii) Observable canonical form (iii) Diagonal form. [7]

5. a) Draw the Jury's table, write its necessary and sufficient conditions. [7]
   b) Consider the following characteristic equation
   $F(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$, Determine whether or not any of the roots of the characteristic equation lie outside the unit circle in the z-plane. Use modified Routh's stability criterion. [7]

6. a) Write design procedure in the w-plane. [7]

   b) A unity feedback system is characterized by the open loop transfer function

$$G_{h0}G(z) = \frac{0.2385(z + 0.8760)}{(z - 1)(z - 0.2644)}$$

   The sampling period T=0.2 sec, Determine steady state errors for following (i) Unit Step (ii) Unit ramp (iii) Unit Parabolic. [7]

7. a) Derive the Ackermann's formula for state feedback gain matrix. [4]

   b) Consider the system

$$X(k+1) = GX(k) + Hu(k)$$

$$G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.12 & -0.01 & 1 \end{bmatrix}; H = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   Determine a suitable state feedback gain matrix 'K' such that the system will have the closed loop poles at 0.3, 0.4, 0.6. [10]

Code No: **R1642021** **R16** **Set No. 2**

IV B.Tech II Semester Regular Examinations, September - 2020

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any FOUR questions from Part-B

*****

PART–A *(14 Marks)*

1. a) Write a statement of sampling theorem. [2]
   b) What is the z-transform of te^{-at}? [2]
   c) Explain the concept of observability. [2]
   d) Write about the primary strips and complementary strips with neat schematic. [2]
   e) Derive an expression for steady state error for step input. [3]
   f) Write statement on sufficient condition for design of state feedback controller through pole placement. [3]

PART–B *(4x14 = 56 Marks)*

2. a) Derive the transfer function of zero order hold. [7]
   b) Explain the block diagram representation of the sample and hold devices. [7]

3. For the sampled data system as shown in figure.3 given below, find (i) Pulse transfer function $\frac{Y(z)}{R(z)}$ (ii) Output $y(k)\ for\ r(t) = unit\ step\ (t = 1\ sec)$.

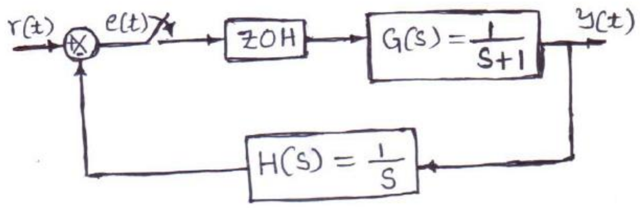


Figure.3 [14]

4. a) Consider the following system $\frac{y(z)}{u(z)} = \frac{z+1}{z^2+z+0.16}$, Obtain (i) Controllable canonical form (ii) Observable canonical form (iii) Diagonal form. [7]
   b) Consider the following pulse transfer function system
   $$\frac{y(z)}{u(z)} = \frac{z^{-1}(1+0.8z^{-1})}{1+1.3z^{-1}+0.4z^{-2}}$$
   Test the state controllability and observability. [7]

5. a) Consider the following characteristic equation $z^3 + 2.1z^2 + 1.44z + 0.32 = 0$, Determine whether or not any of the roots of the characteristic equation lie outside the unit circle centered at the origin of the z-plane. [7]
   b) Determine the stability of the following discrete time system
   $$\frac{y(z)}{x(z)} = \frac{z^{-3}}{1+0.5z^{-1}-1.34z^{-2}+0.24z^{-3}}$$ [7]

6. a) Write about the general rules for constructing Root Loci. [7]

   b) The feed forward pulse transfer function is given

$$G(z) = \frac{Kz(1 - e^{-T})}{(z-1)(z - e^{-T})}$$

   Investigate the effect of the sampling period T on the steady state accuracy of the unit ramp response for the following (i) T=0.5 Sec, K=2 (ii) T=1 Sec, K=2 (iii) T=2 Sec, K=2. Write comment on the above cases. [7]

7. a) Derive necessary condition for the design of state feedback controller through pole placement. [7]

   b) A regulator system has the plant

$$X(k+1) = \begin{pmatrix} 0 & 1 \\ -0.16 & -1 \end{pmatrix} X(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

   Design a full order state observer, the desired eigen values of the observer matrix are -1.8-j2.4, -1.8+j2.4. [7]

Code No: **R1642021** **R16** **Set No. 3**

IV B.Tech II Semester Regular Examinations, September - 2020

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any FOUR questions from Part-B
*****

PART–A *(14 Marks)*

1. a) Write the DCS example of a digital computer controlled rolling mill regulating system. [3]
   b) What is the z-transform of $\cos\omega t$? [2]
   c) Write the diagonal canonical form. [2]
   d) Investigate the mapping from s-plane to z-plane of the constant frequency loci with neat sketch. [2]
   e) Derive an expression for steady state error for ramp input. [3]
   f) What is the purpose of an observer? [2]

PART–B *(4x14 = 56 Marks)*

2. a) List out the merits of digital systems. [4]
   b) State and explain sampling theorem with neat sketch. [10]

3. a) Solve the difference equation
   $$y(k+2)+3y(k+1)+2y(k)=r(k);$$
   $r(k)=unit\ step, y(0)=1\ and\ y(1)=0$ [7]
   b) Obtain the inverse z-transform of $x(z)=\frac{z^{-2}}{(1-z^{-1})^3}$ [7]

4. a) What are the various methods of evaluation of state transition matrix? Explain any one method. [7]
   b) Obtain the state equation and output equation for the system defined by
   $$\frac{y(z)}{u(z)}=\frac{z^{-1}+5z^{-2}}{1+4z^{-1}+3z^{-2}}$$ [7]

5. a) Write about the modified Routh's stability criterion. [7]
   b) Consider the system described by
   $$y(k)-0.6y(k-1)-0.81y(k-2)+0.67y(k-3)-0.12y(k-4)=x(k)$$
   Where x(k) is the input and y(k) is the output of the system. Determine the stability of the system by using Jury's stability criterion. [7]

6. Consider the system as shown in figure.6. Assume that the digital controller is of the integral type.

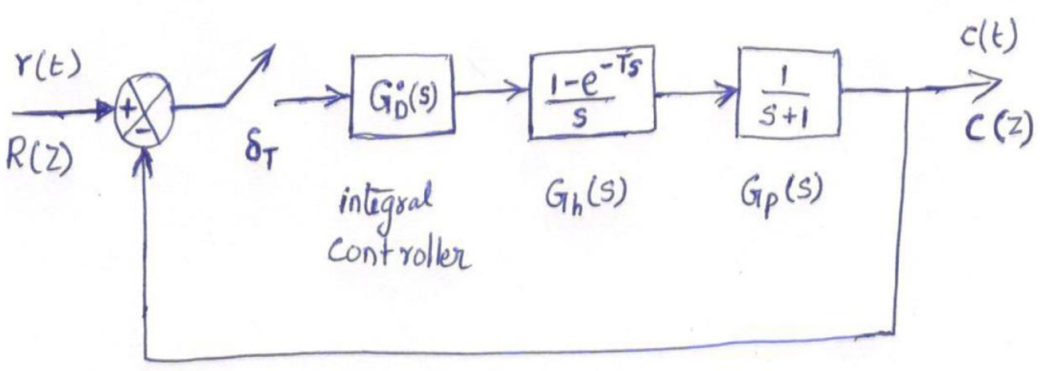


Figure.6

Draw root locus diagram for the system of the sampling period T=0.5. Also determine the critical value of K for T=0.5. Locate the closed loop poles corresponding to K=2 for T=0.5. [14]

7. a) Derive sufficient condition for the design of state feedback controller through pole placement. [4]

b) Consider the system is given by

$$X(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

Determine a suitable state feedback gain matrix 'K' to place the eigen values at 0.5, 0.6, 0.7. [10]

Code No: **R1642021** **R16** **Set No. 4**

IV B.Tech II Semester Regular Examinations, September - 2020

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any FOUR questions from Part-B

*****

PART–A *(14 Marks)*

1. a) Enumerate advantages of digital systems. [2]
   b) Define z-transform and write z transform of unit step function. [2]
   c) Write the Jordan canonical form. [2]
   d) Write comment on the stability of $P(z) = z^2 - 0.25 = 0$ by using modified Routh's stability criterion. [3]
   e) Derive an expression for steady state error for parabolic input. [3]
   f) What is reduced order observer? [2]

PART–B *(4x14 = 56 Marks)*

2. Draw and explain the configuration of the basic digital control systems with neat block diagram. [14]

3. a) State and explain the initial value and final value theorem. [7]
   b) Using the inversion integral method, obtain the inverse z-transform of $x(z) = \frac{10}{(z-1)(z-2)}$; for k=0,1,2,3…….. [7]

4. a) Obtain the state and output equation of discretization of continuous time state equation. [7]
   b) Obtain the state transition matrix of the following discrete time system

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k)$$

Where

$$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

[7]

5. a) Investigate the mapping between the s-plane and the z-plane with neat schematic. [7]
   b) Consider the discrete time unity feedback control system (T=1 sec) whose open loop pulse transfer function is given by $G(z) = \frac{K(0.3679z+0.2642)}{(z-0.3679)(z-1)}$. Determine the range of K for stability by use of the Jury's stability test. [7]

|''|''|||''|'''|||'|

6. Consider the digital control system shown in figure.6. Design a digital controller in the w-plane such that the phase margin is 50^0, the gain margin is at least 10 dB, and the static velocity error constant K_v is 2 sce^{-1}. Assume that the sampling period is 0.2 sec.

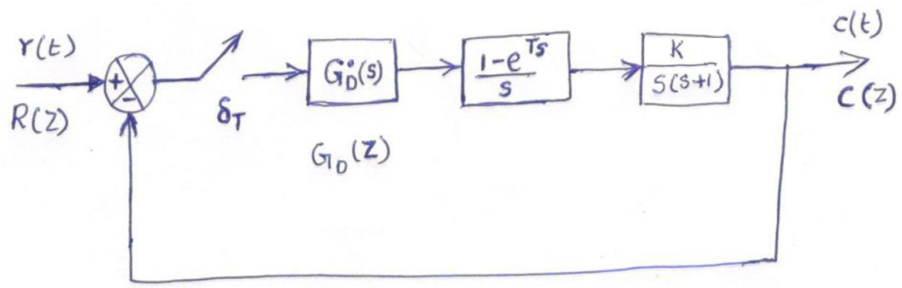


Figure.6

7. a) Explain the full order observer with neat block diagram and also write its error dynamics of the full order state observer. [7]

   b) Consider the system is given by

$$X(k+1) = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} X(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k)$$

   Obtain the state feedback gains 'K' to place the eigen values at 0.1, 0.2 using Ackermann''s formula. [7]

Code No: **RT42021** **R13** **Set No. 1**

IV B.Tech II Semester Regular/Supplementary Examinations, April/May - 2019

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any THREE questions from Part-B

*****

PART–A (22 Marks)

1\. a) Briefly explain the basic components of a digital control system. [4]
   b) What is shifting theorem of z-transforms? [3]
   c) What are the advantages of state space approach compared to conventional approach in system analysis? [4]
   d) What are Primary strips and Complementary Strips? [4]
   e) Explain the need for compensation in digital control systems. [3]
   f) What is pole placement by state feedback? [4]

PART–B (3x16 = 48 Marks)

2\. a) Explain the merits and demerits of digital control systems compared to analog control systems. [8]
   b) Derive the transfer function of zero order hold device. [8]

3\. a) Obtain the pulse transfer function of the system $G(s) = \frac{1-e^{-Ts}}{s}\left(\frac{1}{s(s+1)}\right)$. [8]
   b) Find the inverse Z-Transform of the following:
   $(i)\ F(z) = \frac{z-4}{(z-1)(z-2)^2}$ (ii) $F(z) = \frac{z^2}{(z-1)(z-0.2)}$. [8]

4\. a) Obtain the state transition matrix of the following discrete time systems
$$X(k+1) = GX(k) + Hu(k)$$
$$where\ G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ [8]
   b) Consider the following pulse transfer function system:
$$\frac{Y(z)}{U(z)} = \frac{z^{-1}(1+0.8z^{-1})}{1+1.3z^{-1}+0.4z^{-2}}$$
   Test the state controllability and observability. [8]

5\. a) Determine the stability of the following characteristic equation by using suitable tests. $z^4-1.7z^3+1.04z^2-0.268z + 0.024 = 0$. [8]
   b) With an example explain the stability analysis using Modified routh's stability criterion. [8]

6\. a) The characteristics equation of a discrete time system is $1 + \frac{Kz(1-e^{-T})}{(z-1)(z-e^{T})}=0$, Draw the root locus for T=0.5 sec. [8]
   b) Explain the transient response specifications with reference to unit step response of discrete time response. [8]

7. a) Consider the following system

$$X(k+1) = GX(k) + Hu(k)$$
$$where\ G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Determine a state feedback controller K to place the closed loop poles at z=0.5±j0.5. [8]

b) What is the necessary and sufficient condition for arbitrary pole-placement? Prove the sufficiency of the condition. [8]

Code No: **RT42021** **R13** **Set No. 2**

IV B.Tech II Semester Regular/Supplementary Examinations, April/May - 2019

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any THREE questions from Part-B
*****

PART–A *(22 Marks)*

1. a) What are the disadvantages of digital control systems over analog systems? [4]
   b) Find the inverse Z-transform of $\dfrac{az}{(z-a)^2}$ [3]
   c) What are the properties of State transition matrix? [4]
   d) Distinguish between Routh's criterion and Modified Routh's stability criterion. [4]
   e) List out the steady state specifications. [3]
   f) How is state feedback controller useful for pole placement? [4]

PART–B *(3x16 = 48 Marks)*

2. a) Describe any two examples of digital control system. [8]
   b) Explain the Frequency domain characteristics of zero order hold. [8]

3. a) State and explain the following theorems of z-transforms:
   (i) Initial value theorem (ii) Final Value theorem [8]
   b) The pulse transfer function of digital control systems is given by
   $$G(z) = \frac{5z}{z^2 + 3z + 2}$$
   Find the complete solution to a unit step input and assume that, the initial conditions are zero. [8]

4. a) Consider the system defined by
   $$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$
   $$y(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$
   Determine the conditions on a,b,c and d for complete state controllability and complete observability. [8]
   b) Obtain the state transition matrix of the following discrete time system:
   $$x(k+1) = Gx(k) + Hu(k)$$
   $$y(k) = Cx(k)$$
   Where
   $$G = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = [1 \quad 0]$$ [8]

5. a) Explain the mapping between S-plane and Z-plane. [8]
   b) Write down the rules in Jury stability criterion. [8]

6. a) Explain the design procedure for Lag –Lead compensator in ω-plane. [8]
   b) Explain the angle and magnitude conditions for the characteristic equation 1+G(z)H(z)=0 for drawing root locus. [8]

7. a) Derive 'Ackerman's formula' for pole placement. [8]
   b) Consider the following system

$$X(k+1) = GX(k) + Hu(k)$$
$$where\ G = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}$$

   Determine a state feedback controller K to place the closed loop poles at z=0.6±j0.4. [8]

Code No: **RT42021** **R13** **Set No. 3**

IV B.Tech II Semester Regular/Supplementary Examinations, April/May - 2019

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any THREE questions from Part-B
*****

PART–A *(22 Marks)*

1. a) Write down the advantages of digital control systems over analog systems. [4]
   b) Obtain the z-transform of sin ωt. [3]
   c) Explain the concept of observability. [4]
   d) Write the mapping points between S-Plane and Z-plane. [4]
   e) Write the general form of transfer functions for (i) Lead compensator and (ii) Lag compensator. [3]
   f) Draw the block diagram of a closed loop discrete time system that uses state feedback controller for pole placement. [4]

PART–B *(3x16 = 48 Marks)*

2. a) Draw and explain the general block diagram of discrete data control system. [8]
   b) Explain how a zero order hold helps in data reconstruction. [8]

3. a) Using z-transforms solve the equation given below
   $x(k+2) + 3x(k+1) + 2x(k) = 0, x(0) = 0, x(1) = 1$ [8]
   b) Explain the procedure for obtaining the pulse transfer function of a closed loop transfer function. [8]

4. a) What is state transition matrix? What are its properties? [6]
   b) Given the following state model of the system
   $$X(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} X(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} U(k)$$
   $$Y(k) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} X(k)$$
   Obtain the state transition matrix. [10]

5. a) Examine the stability of the following characteristic equation using jury stability analysis. $P(Z)=Z^4-1.2Z^3+0.07Z^2+0.3Z-0.08 = 0$ [8]
   b) Explain stability analysis using bilinear transformation and Routh stability criterion. [8]

6. a) Explain the design procedure in the ω - plane of lead compensator. [8]
   b) A block diagram of a digital control system is shown in figure, Draw the root locus for sampling period T=0.4 sec.

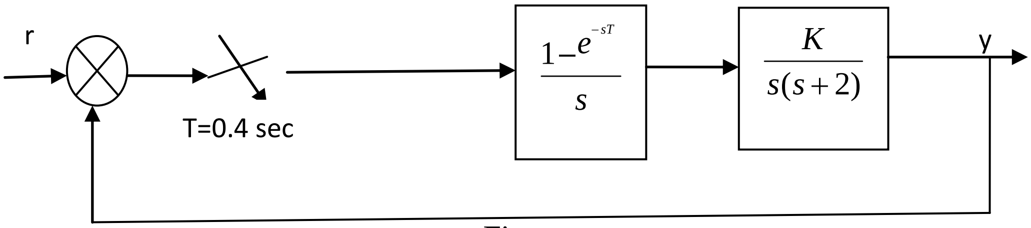


Figure [8]

7. a) State and prove the necessary condition for arbitrary pole-placement? [8]
   b) Consider the following system

$$X(k+1) = GX(k) + Hu(k)$$
$$where\ G = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}$$

Determine a state feedback controller K to place the closed loop poles at z=0.4±j0.6. [8]

Code No: **RT42021** **R13** **Set No. 4**

IV B.Tech II Semester Regular/Supplementary Examinations, April/May - 2019

DIGITAL CONTROL SYSTEMS

(Electrical and Electronics Engineering)

Time: 3 hours **Max. Marks: 70**

Question paper consists of Part-A and Part-B
Answer ALL sub questions from Part-A
Answer any THREE questions from Part-B
*****

PART–A ***(22 Marks)***

1. a) What is sampling theorem? What is its importance? [4]
   b) State initial and final value theorems. [4]
   c) Explain the concept of controllability. [3]
   d) Explain the mapping between S-plane and Z-plane. [4]
   e) Write the expressions for static position error constant and steady state error in response to a unit step input in discrete time systems. [4]
   f) Explain 'Ackerman's formula' for pole placement. [3]

PART–B ***(3x16 = 48 Marks)***

2. a) Explain in detail the process of sampling and reconstruction of signals. [8]
   b) Draw the schematic diagram of basic discrete data control system and explain the same. [8]

3. a) Find inverse z –transform of (i) $\frac{1}{(z+a)^2}$ (ii) $\frac{2}{(2z-1)^2}$ [8]
   b) Explain the procedure for obtaining the pulse transfer function of open loop transfer function. [8]

4. a) Derive an expression to find the state transition matrix of a discrete system. [8]
   b) Obtain the discrete time state and output equations of the following continuous time system. $\dot{X} = AX + bu$; $Y = CX$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $C = [1 \quad 0]$ [8]

5. a) Explain the mapping procedure for the following from s-plane to z-plane
   (i) The constant damping loci (ii) The constant frequency loci [8]
   b) Use the Routh-Hurwitz criterion to find the stable range of K for the closed loop unity feedback system with loop gain $F(z) = \dfrac{K(z-1)}{(z-0.1)(z-0.8)}$. [8]

6. Consider the digital control system shown in figure, where the plant transfer function is $\frac{1}{s^2}$. Design a digital controller in the w-plane such that the phase margin is 50^0 and the gain margin is atleast 10 dB. The sampling period is 0.1 sec.

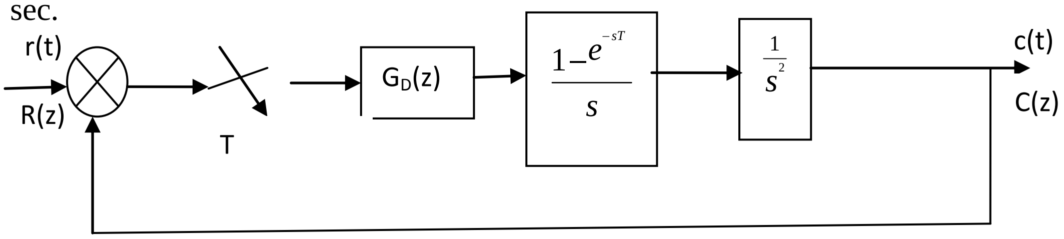


Figure [16]

7. a) Explain the step by step procedure of pole placement by state feedback in discrete systems. [8]

   b) Consider the following system

$$X(k+1) = GX(k) + Hu(k)$$
$$where\ G = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}$$

   Determine a state feedback controller K to place the closed loop poles at z=0.3±j0.3. [8]